Worksheet 6: Quotients

1. For the groups and subgroups given below, write out the elements of $G/H$. Come up with a description of all elements in a given coset.

For example: $G = \mathbb{Z}$, $H = 3\mathbb{Z}$. The elements of the group $G/H$ are the cosets of $H$, which are

\[
H = \{\ldots, -3, 0, 3, 6, 9, \ldots\}
\]

\[
1 + H = \{\ldots, -2, 1, 4, 7, 10, \ldots\}
\]

\[
2 + H = \{\ldots, -1, 2, 5, 8, 11, \ldots\}
\]

A description is: the elements in the coset $a + H$ are the integers that have a remainder of $a$ when you divide by 3.

(a) $G = \mathbb{Z}$, $H = 4\mathbb{Z}$.

(b) $G = \mathbb{Z}_{10}$, $H = \langle 5 \rangle = \{0, 5\}$.

(c) $G = \mathbb{Z} \times \mathbb{Z}$, $H = \langle (1, 1) \rangle$.

(d) $G = D_3$, $H = \langle r \rangle$.

\[\text{If you have extra time, you should justify why each } H \text{ is normal.}\]
2. For the homomorphisms given below, determine ker $\phi$.

FOR EXAMPLE: $\phi : \mathbb{Z} \to \mathbb{Z}_3$ given by $\phi(n) = n \mod 3$. The kernel is the set of elements such that $\phi(n) = 0$, meaning $n = 0 \mod 3$, meaning $n$ is a multiple of 3. Therefore, ker $\phi = 3\mathbb{Z}$.

(a) $\phi : \mathbb{Z} \to \mathbb{Z}_4$ given by $\phi(n) = n \mod 4$.

(b) $\phi : \mathbb{Z}_{10} \to \mathbb{Z}_5$ given by $\phi(n) = n \mod 5$.

(c) $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $\phi(n, m) = n - m$.

3. We proved in class that, for a homomorphism $\phi : G \to G'$, $G/\ker \phi \cong \phi[G]$. If $\phi$ is onto, this says that $G/\ker \phi \cong G'$. Match the kernels in problem 2(a)(b)(c) with the subgroups in problem 1 and use your description to explain the isomorphism.

FOR EXAMPLE: The elements of $\mathbb{Z}/3\mathbb{Z}$ are the cosets $H$, $1 + H$, and $2 + H$. The coset $a + H$ represents all integers that are equal to $a \mod 3$, so the isomorphism $\mathbb{Z}/\ker \phi \cong \mathbb{Z}_3$ matches the coset $a + H$ to the remainder of the elements modulo 3.

4. For part 1(d), you saw that $G/H$ had two elements. There is only one group with two elements, $\mathbb{Z}_2$, meaning there should be an isomorphism $G/H \cong \mathbb{Z}_2$. Can you find a homomorphism $\phi : D_3 \to \mathbb{Z}_2$ with kernel $H$?

\[2\text{If you have extra time, you should justify why each function is a homomorphism.}\]