Worksheet 6: Quotients: Solutions

1. For the groups and subgroups given below, write out the elements of G/H[1]. Come up with a description of all elements in a given coset.

   FOR EXAMPLE: $G = \mathbb{Z}$, $H = 3\mathbb{Z}$. The elements of the group G/H are the cosets of H, which are

$$\begin{aligned} H &= \{\dots, -3, 0, 3, 6, 9, \dots\} \\ 1 + H &= \{\dots, -2, 1, 4, 7, 10, \dots\} \\ 2 + H &= \{\dots, -1, 2, 5, 8, 11, \dots\} \end{aligned}$$

   A description is: the elements in the coset $a + H$ are the integers that have a remainder of a when you divide by 3.

   (a) $G = \mathbb{Z}$, $H = 4\mathbb{Z}$.

Solution.

$$\begin{aligned} H &= \{\dots, -4, 0, 4, 8, 12, \dots\} \\ 1 + H &= \{\dots, -3, 1, 5, 9, 13, \dots\} \\ 2 + H &= \{\dots, -2, 2, 6, 10, 14, \dots\} \\ 3 + H &= \{\dots, -1, 3, 7, 11, 15, \dots\} \end{aligned}$$

The elements in the coset $a + H$ are the integers that have a remainder of a when you divide by 4.

   (b) $G = \mathbb{Z}_{10}$, $H = \langle 5 \rangle = \{0, 5\}$.

Solution.

$$\begin{aligned} H &= \{0, 5\} \\ 1 + H &= \{1, 6\} \\ 2 + H &= \{2, 7\} \\ 3 + H &= \{3, 8\} \\ 4 + H &= \{4, 9\} \end{aligned}$$

The elements in the coset $a + H$ are the elements of $\mathbb{Z}_{10}$ that have a remainder of a when you divide by 5.

   (c) $G = \mathbb{Z} \times \mathbb{Z}$, $H = \langle (1, 1) \rangle$.

[1] If you have extra time, you should justify why each H is normal.

Solution.

$$\begin{aligned}
H &= \{\dots, (-1,-1), (0,0), (1,1), (2,2), \dots\} \\
(1,0) + H &= \{\dots, (0,-1), (1,0), (2,1), (3,2), \dots\} \\
(2,0) + H &= \{\dots, (1,-1), (2,0), (3,1), (4,2), \dots\} \\
(3,0) + H &= \{\dots, (2,-1), (3,0), (4,1), (5,2), \dots\} \\
&\vdots \\
(n,0) + H &= \{\dots, (n-1,-1), (n,0), (n+1,1), (n+2,2), \dots\} \\
&\vdots
\end{aligned}$$

The elements in the coset $(a,0)+H$ are all pairs $(n,m) \in \mathbb{Z}\times\mathbb{Z}$ such that $n-m=a$.

(d) $G = D_3$, $H = \langle r \rangle$.

Solution. Recall that $D_3 = \{e, r, r^2, f, fr, fr^2\}$ subject to the relations $r^3 = e, f^2 = e$, and $rf = fr^2$.

Then, $H = \langle r \rangle = \{e, r, r^2\}$. So, the elements of G/H are

$$\begin{aligned}
H &= \{e, r, r^2\} \\
fH &= \{f, fr, fr^2\}
\end{aligned}$$

Thinking about r and f as symmetry operations on the triangle, the elements of H are those that leave the triangle facing up (in any rotation) and the elements of fH are those that flip the triangle over (in any rotation).

2. For the homomorphisms[2] given below, determine $\ker\phi$.

FOR EXAMPLE: $\phi : \mathbb{Z} \to \mathbb{Z}_3$ given by $\phi(n) = n \mod 3$. The kernel is the set of elements such that $\phi(n) = 0$, meaning $n = 0 \mod 3$, meaning n is a multiple of 3. Therefore, $\ker\phi = 3\mathbb{Z}$.

(a) $\phi : \mathbb{Z} \to \mathbb{Z}_4$ given by $\phi(n) = n \mod 4$.

Solution.

$$\begin{aligned}
\ker\phi &= \{n \in \mathbb{Z} \mid \phi(n) = 0\} \\
&= \{n \in \mathbb{Z} \mid n = 0 \mod 4\} \\
&= 4\mathbb{Z}.
\end{aligned}$$

(b) $\phi : \mathbb{Z}_{10} \to \mathbb{Z}_5$ given by $\phi(n) = n \mod 5$.

Solution.

$$\begin{aligned}
\ker\phi &= \{n \in \mathbb{Z}_{10} \mid \phi(n) = 0\} \\
&= \{n \in \mathbb{Z}_{10} \mid n = 0 \mod 5\} \\
&= \{0, 5\}.
\end{aligned}$$

[2] If you have extra time, you should justify why each function is a homomorphism.

(c) $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $\phi(n, m) = n - m$.

Solution.

$$\begin{aligned}\ker \phi &= \{(n,m) \in \mathbb{Z} \times \mathbb{Z} \mid \phi(n,m) = 0\} \\ &= \{(n,m) \in \mathbb{Z} \times \mathbb{Z} \mid n - m = 0\} \\ &= \{(n,m) \in \mathbb{Z} \times \mathbb{Z} \mid n = m\} \\ &= \{\dots, (-1,-1), (0,0), (1,1), (2,2), \dots\} \\ &= \langle (1,1) \rangle.\end{aligned}$$

3. We proved in class that, for a homomorphism $\phi : G \to G'$, $G/\ker\phi \cong \phi[G]$. If ϕ is onto, this says that $G/\ker\phi \cong G'$. Match the kernels in problem 2(a)(b)(c) with the subgroups in problem 1 and *use your description* to explain the isomorphism.

   FOR EXAMPLE: The elements of $\mathbb{Z}/3\mathbb{Z}$ are the cosets H, $1 + H$, and $2 + H$. The coset $a + H$ represents all integers that are equal to $a \mod 3$, so the isomorphism $\mathbb{Z}/\ker\phi \cong \mathbb{Z}_3$ matches the coset $a + H$ to the remainder of the elements modulo 3.

Solution. For part 2(a), we have $\ker\phi = 4\mathbb{Z}$, so it matches H in part 1(a), and the coset $a + H$ in G/H represents all integers that are equal to $a \mod 4$, so the isomorphism $\mathbb{Z}/\ker\phi \cong \mathbb{Z}_4$ matches the coset $a + H$ with a, the remainder of the elements modulo 4.

For part 2(b), we have that $\ker\phi = \langle 5 \rangle$, so it matches H in part 1(b). The coset $a + H$ represents all elements in $\mathbb{Z}_{10}$ that are equal to $a \mod 5$, so the isomorphism $\mathbb{Z}_{10}/\ker\phi \cong \mathbb{Z}_5$ matches the coset $a + H$ with a, the remainder mod 5.

For part 2(c), we have that $\ker\phi = \langle (1,1) \rangle$, so it matches H in part 1(c). The cosets from part 1(c) were $(a, 0) + H$, representing all elements (n, m) such that $n - m = a$. So, the isomorphism $\mathbb{Z} \times \mathbb{Z}/\ker\phi \cong \mathbb{Z}$ matches the coset $(a, 0) + H$ with the difference $a = n - m$ for any element $(n, m) \in (a, 0) + H$.

4. For part 1(d), you saw that G/H had two elements. There is only one group with two elements, $\mathbb{Z}_2$, meaning there should be an isomorphism $G/H \cong \mathbb{Z}_2$. Can you find a homomorphism $\phi : D_3 \to \mathbb{Z}_2$ with kernel H?

Solution. Every element in D_3 is of the form $f^i r^j$, where $i \in \{0, 1\}$ and $j \in \{0, 1, 2\}$. We can define a homomorphism $\phi : D_3 \to \mathbb{Z}_2$ by $\phi(f^i r^j) = i$. You *need to check that this is a homomorphism.* Once you do that, the kernel is all elements such that $i = 0$, so the kernel is all elements of the form r^j for any $j \in \{0, 1, 2\}$, so $\ker\phi = H$. This also matches the description above: if $i = 0$, the triangle has not been flipped over. If $i = 1$, the triangle has been flipped over. So, the isomorphism $D_3/H \cong \mathbb{Z}_2$ matches the coset of elements where the triangle has not been flipped with 0 and it matched the coset of elements where the triangle has been flipped with 1.