Math 100A: Abstract Algebra I (UC San Diego, fall 2017)
Homework 5: due November 8 at 5pm
Reminder: no homework due November 15, as there is a midterm that day.

(1) Artin, Chapter 2, exercise 11.3.
(2) Artin, Chapter 2, exercise 11.6.
(3) Artin, Chapter 2, exercise 12.4. (Hint: the answer to the last question is yes!)
(4) Artin, Chapter 2, exercise 12.5.
(6) Artin, Chapter 6, exercise 4.1.
(7) Artin, Chapter 6, exercise 4.2.
(8) Prove that for any positive integers $r, s$, there is an isomorphism of groups

$$(\mathbb{Z}/r\mathbb{Z})^+ \times (\mathbb{Z}/s\mathbb{Z})^+ \cong (\mathbb{Z}/\gcd(r, s)\mathbb{Z})^+ \times (\mathbb{Z}/\lcm(r, s)\mathbb{Z})^+.$$  

(Hint: one option is to factor both sides into prime powers using the Chinese remainder theorem stated in class.)

(9) Let $p$ be a prime number. In this exercise, we take advantage of the fact that under arithmetic mod $p$, every nonzero value has a multiplicative inverse. (That is, the nonzero residue classes mod $p$ form a group under multiplication.)

(a) Let $n$ be a positive integer. Prove that the set of $n \times n$ matrices with entries in $\mathbb{Z}/p\mathbb{Z}$ having nonzero determinant form a group. I’ll call this $\text{GL}_n(\mathbb{Z}/p\mathbb{Z})$. (You might find it helpful to read section 3.2 in the text first.)

(b) Compute the order of $\text{GL}_2(\mathbb{Z}/p\mathbb{Z})$. Optional: do the same for $\text{GL}_n(\mathbb{Z}/p\mathbb{Z})$.

(10) For any group $G$, an automorphism of $G$ is an isomorphism of $G$ with itself.

(a) Prove that the automorphisms of $G$ form another group. This group is usually denoted $\text{Aut}(G)$.

(b) Let $V$ be the Klein four-group. Prove that $\text{Aut}(V)$ is isomorphic to $S_3$. Optional: also show that $\text{Aut}(V) \cong \text{GL}_2(\mathbb{Z}/2\mathbb{Z})$, and use this to check your formula from the previous exercise.