In addition to this assignment, you might find it helpful to look at the optional homework 6.5, in which some additional examples of Galois groups are given.

No office hours Wednesday, May 30. However, if you want to meet by videoconference, let me know and I will set that up. (And no lecture Monday, May 28 because of the Memorial Day holiday.)

Reminder: CAPE evaluations can be done online anytime until Monday, June 11 at 8am.

(1) Artin, chapter 16, exercise 8.2(b,c).
(2) Artin, chapter 16, exercise 9.8(a,b,c).
(3) Artin, chapter 16, exercise 9.12(a,b,e,f). The next exercise is a variant of (d).
(4) Let \( f(x) \) be the polynomial \( x^4 + x - 1 \).
   (a) Note that \( f(2) = 13, f(0) = -1, f(2) = 17 \). Using the intermediate value theorem, show that \( f \) has at least two real roots.
   (b) Using Rolle’s theorem, show that \( f \) has exactly two real roots. Deduce that the Galois group of \( f \) contains a transposition.
   (c) It turns out that the discriminant of \( f \) is 229. Deduce that the Galois group of \( f \) is \( S_4 \).
   (d) Do Artin, chapter 16, exercise 9.9(a), then explain how it yields the same conclusion.
(5) Artin, chapter 16, exercise 12.8.
(6) (a) Let \( K \) be the splitting field of the polynomial \( x^5 - 2 \). Prove that \( K/\mathbb{Q} \) is a Galois extension of degree 20, and describe its Galois group as a subgroup of \( S_5 \).
   (b) Artin, chapter 16, exercise 12.2. (Hint: you should get five answers up to conjugation: \( C_5, D_5, A_5, S_5, \) and the group from (a).)
(7) Let \( K/F \) be a Galois extension with group \( G \). Let \( H \) be a subgroup of \( G \).
   (a) Prove that \( y \) is a primitive element of \( K^H \) over \( F \) if and only if \( g(y) \neq y \) for all \( y \in G - H \). (Hint: apply the Galois correspondence to the intermediate field \( K(y) \).)
   (b) A normal basis of \( K \) over \( F \) is a basis which is also an orbit for the action of \( G \) on \( K \). Show that if \( x \in K \) belongs to a normal basis, then \( y := \sum_{h \in H} h(x) \) is a primitive element of \( K^H \) over \( F \).
   (c) For \( F := \mathbb{Q}, K := \mathbb{Q}(\zeta_n) \) with \( n \) squarefree, it turns out that the primitive \( n \)-th roots of unity in \( K \) form a normal basis. (This was discussed on a previous homework.) Using this fact, construct an element \( \alpha \in \mathbb{C} \) such that \( \mathbb{Q}(\alpha) \) is Galois over \( \mathbb{Q} \) with group \( C_3 \times C_3 \times C_3 \).
(8) Artin, chapter 16, exercise 10.4. You may find the previous exercise helpful.
(9) Define the polynomial \( f(x) = x^5 - 6x + 3 \).
   (a) Prove that \( f \) is irreducible over \( \mathbb{Q} \). (Hint: use the Eisenstein criterion.)
   (b) Prove that in \( \mathbb{C}, f \) has three real roots and two nonreal roots. (Hint: the sign changes among \( f(-2), f(0), f(1), f(2) \) guarantee three real roots. If there were more, what would Rolle’s theorem imply about the roots of \( f' \)?)
   (c) Deduce that the splitting field of \( f \) has Galois group \( S_5 \), and therefore \( f \) is not solvable in radicals.
(10) Artin, chapter 16, exercise M.12.