Instructions:

You have 50 minutes. No textbooks, notes, or outside resources are allowed. Use the space provided for your answers. If you need more space, you may use the back of the page.

The scope of the midterm is homeworks 3–5. In particular, you will not be asked to make any computations using character tables, but theoretical properties of group representations are fair game.

I will be having extra office hours Friday, May 11, 4-6pm in APM 7202.

Throughout, \( \zeta_n \) means \( \exp(2\pi i/n) \in \mathbb{C} \).

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**Problem 1. [10 points.]**

Give a one-sentence answer to each of the following.

(a) For \( \rho : G \to \text{GL}(V) \) an irreducible representation, give the name and statement of the result describing the \( G \)-invariant transformations \( T : V \to V \).

Schur’s lemma states that every \( G \)-invariant transformation \( T : V \to V \) is multiplication by a scalar.
(b) For $F \rightarrow K$ a field extension, explain why the minimal polynomial of an element $\alpha \in K$ over $F$ is irreducible.

The minimal polynomial $P(x)$ of $\alpha$ is the smallest-degree polynomial satisfying $P(\alpha) = 0$; if it were to factor nontrivially as $Q_1Q_2$, then either $Q_1(\alpha) = 0$ or $Q_2(\alpha) = 0$ and this is a contradiction.

(c) For $P(x) \in \mathbb{Z}[x]$ for which every root of $P(x)$ in $\mathbb{C}$ has absolute value 1, describe all possible irreducible factors of $P(x)$ over $\mathbb{Q}$.

By Kronecker’s theorem, every irreducible factor must be a cyclotomic polynomial.

Problem 2. [5 points.]

For $n$ an odd integer, state (with justification) the relationship between the cyclotomic polynomials $\Phi_n(x)$ and $\Phi_{2n}(x)$.

The roots of $\Phi_n(x)$ are the elements of $\mathbb{C}^\times$ of order $n$; for each such root $\alpha$, $-\alpha$ has order $2n$. The roots of $\Phi_{2n}(x)$ are the elements of $\mathbb{C}^\times$ of order $2n$; for each such root $\alpha$, $-\alpha$ has order $n$. By matching the leading coefficients, we get $\Phi_{2n}(x) = -\Phi_n(-x)$.

Problem 3. [10 points.]

Compute the degrees of the following field extensions. No further justification is needed.

(a) $\mathbb{Q} \rightarrow \mathbb{Q}(2^{1/5})$. This degree is 5.

(b) $\mathbb{Q} \rightarrow \mathbb{Q}(\zeta_{18})$. This degree is $\varphi(18) = 2^1(2 - 1)3^2 - 1(3 - 1) = 6$.

(c) $\mathbb{Q}(\sqrt{6}) \rightarrow \mathbb{Q}(\sqrt{2} + \sqrt{3})$. The degree is 2. (As shown in class, $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ has degree 4 over $\mathbb{Q}$.)

Problem 4. [10 points.]

For each of the following, state whether or not the following ruler-and-compass construction is possible or impossible. If possible, do nothing else; if impossible, justify your answer.

(a) Given distinct points $p_0, p_1, p_2$, construct a point $p_3$ such that the measure of $\angle p_0p_1p_3$ is one-fourth that of $\angle p_0p_1p_2$. (Here “possible” means that this can be done no matter what the initial points are.) Possible. [One can bisect any angle, so doing this twice does the job.]

(b) Given the endpoints $p_0, p_1$ of a line segment of length 1, construct a segment of length $2^{1/3}$. Impossible: the degree $[\mathbb{Q}(2^{1/3}) : \mathbb{Q}]$ is 3, which is not a power of 2.

(c) Given points $p_0, p_1$, construct a regular 13-gon with one side equal to $p_0p_1$. Impossible: the degree $[\mathbb{Q}(\zeta_{13}) : \mathbb{Q}] = 12$ is not a power of 2. [Here we are using a fact I stated but didn’t prove in class: the point $(x, y)$ is constructible if and only if the complex number $x + iy$ appears in some chain of quadratic extensions.]
(d) Given points $p_0, p_1, p_2$ such that the segment $p_0p_1$ has length 1 and the segment $p_1p_2$ has length $\cos \frac{2\pi}{7}$, construct a segment of length $\cos \frac{\pi}{7}$. Possible. [Given $\cos \frac{2\pi}{7}$, one can construct $\sin \frac{2\pi}{7}$ and hence any real number in $\mathbb{Q}(\zeta_7)$. Since $\mathbb{Q}(\zeta_7) = \mathbb{Q}(\zeta_{14})$, this includes $\cos \frac{\pi}{7}$.]