Lattices

PS2 is posted (due Oct 22).
Lattices in rational vector spaces

$V = \text{finite dim. } \mathbb{Q}\text{-vector space}$

$L \subset V$ is a lattice if it is free.

$L$ is a finitely generated $\mathbb{Z}$-submod.

which spans $V$ over $\mathbb{Q}$.

eg. for $K$ a number field, $\mathcal{O}_K$ is a lattice in $K$. 

"6dcl"
Lattices in Euclidean spaces

$V$ is a finite-dim $\mathbb{R}$-vec space with a positive-definite inner product.

A lattice $L$ in $V$ is a full-rank $\mathbb{Z}$-submodule of $V$ whose $\mathbb{R}$-span is $V$.

$\text{vol}(L) \subseteq \mathbb{R}$ (so not $2 + 2\sqrt{2} \in \mathbb{R}$).

$L \subseteq V$ is compact.
Lattices in the wider world

- Chemistry and materials
- Telecommunications/coding theory
- Cryptography (especially post-quantum)

Conway & Sloane
Sphere packings,
Lattices & graphs (SL4/6)
The lattice of a number field: imaginary quadratic case

$$<z_1, z_2> = \Re(z_1 \cdot z_2)$$
The lattice of a number field: real quadratic case

\[ \mathbb{Z}(\sqrt{2}) \subset \mathbb{R} \times \mathbb{R} \]
\[ (a+6\sqrt{2}) \mapsto (a+6\sqrt{2}, a-6\sqrt{2}) \]

```python
l = [(a+b*sqrt(2.0), a-b*sqrt(2.0)) for a in range(-10,10) for b in range(-10,10)]
l = [(x,y) for x,y in l if abs(x) <= 10 and abs(y) <= 10]
list_plot(l, aspect_ratio=1) + plot(1/x, (x,0.1,10), color="red") + plot(-1/x, (x,0.1,10), color="red")
```
The signature of a number field

For \( K \) a number field with \([K:\mathbb{Q}] = n\), the signature of \( K \) is the pair \((r_1, r_2)\):

\[
\begin{align*}
  r_1 &= \text{# of real embeddings: } K \hookrightarrow \mathbb{R} \\
  r_2 &= \text{# of pairs of complex embeddings: } K \hookrightarrow \mathbb{C}
\end{align*}
\]

Note: \( r_1 + 2r_2 = n \)

\( n = 2 \) (e.g., \( \mathbb{Q}(\sqrt{2}) \)), \( \mathbb{Q}(\sqrt{2}) \) is quadratic.
The additive lattice of a number field

\[ j : K \rightarrow K^C = \prod C \]
\[ a \mapsto j(a) = (\tau(a))_C \]

\( K \) carries standard Hermitian inner product
\[ \langle x, y \rangle = \sum \overline{X}_p Y_p \]

\( F : K^C \rightarrow K^C \) complex conjugation
\[ (F(x))_p = \overline{X}_p \]
\[ \overline{T} = \mathcal{O}_T \]

\( K_{1R} = F\text{-invariants of } K^C \).
\[ \phi : K \rightarrow K_{1R}, \text{ restriction map.} \]
The trace pairing revisited
The covolume of a Euclidean lattice

Image is discrete: for \( a \in \mathbb{R}^n \), \( \| J(a) \| = |\text{Norm}(a)| / \alpha \) \( \alpha \neq 0 \)

\( L \leq V \) Euclidean space

inner product \( \rightarrow \) normalization of volume measure

\[ \text{covolume of } L = \text{volume of } V \]

\( \alpha \) is a fundamental domain of \( L \)

\[ = \text{volume of } V / \ell \] for induced measure.
The absolute discriminant as a covolume

Let \( \mathbf{V} \) be a Euclidean space

\[ \mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbf{V} \] be a basis

then covolume = \( \left( \det(\langle \mathbf{x}_i, \mathbf{x}_j \rangle) \right)^{1/2} \)

In case of \( \mathbb{Q} \subset \mathbb{K} \subset \mathbb{R} \),

\[ (\text{covolume})^2 = \text{signature} \]

similar if \( \mathbb{I} = \text{ideal} \)

\[ (\text{covolume of } \mathbb{I})^2 = \text{discriminant of } \mathbb{I} \]

\[ = \text{sign} \mathbb{K} \cdot [\mathbb{K} : \mathbb{I}]^2 \]
E.g., for Gaussian lattice,
\[ |\text{co-volume}|^2 = 4, \quad \text{not } 1. \]
\[ (\text{divc} = -4) \]