For the Chinese remainder theorem for Dedekind domains, see for example Neukirch Theorem I.3.6.

1. Let $R$ be a Dedekind domain. Prove that for every nonzero ideal $I$ of $R$, $R/I$ is a principal ideal ring. (Hint: use the Chinese remainder theorem to reduce to the case where $I$ is a power of a prime $p$. In that case, choose $\pi \in p \setminus p^2$ and consider the powers of $\pi$.)

2. Let $R$ be a Dedekind domain. Prove that every ideal of $R$ can be generated by at most two elements.

3. Let $R$ be the ring $\mathbb{Z}[\alpha]/(\alpha^3 - \alpha + 1)$. Show that the prime factorization of the principal ideal $23R$ is given by
   
   $23R = (23, \alpha - 10)^2(23, \alpha - 3)$.

   (In particular, you should show that the factors are indeed prime.)

4. Prove that the constant $2^n$ in Minkowski’s lattice point theorem cannot be improved.

5. Let $K$ be a number field. Using the finiteness of the class group of $K$, prove that there exists a finite extension $L$ of $K$ such that every ideal of $\mathcal{O}_K$ generates a principal ideal of $\mathcal{O}_L$.

6. Let $p > 2$ be a prime number and put $K = \mathbb{Q}(\zeta_p)$.

   (a) Compute $\text{Trace}_{K/\mathbb{Q}}(\zeta_p^j)$ for $j = 0, \ldots, p - 1$.

   (b) Compute $\text{Norm}_{K/\mathbb{Q}}(1 - \zeta_p)$.

   (c) Show that $(1 - \zeta_p)\mathcal{O}_K \cap \mathbb{Z} = p\mathbb{Z}$. (Note: we will use this later to show that $\mathcal{O}_K = \mathbb{Z}[\zeta_p]$, so don’t assume this here.)