Let $k$ be a finite field of order $q$ and fix an additive character (homomorphism) $\psi: k \to \mathbb{C}^\times$. For $\chi: k^\times \to \mathbb{C}^\times$ a nontrivial multiplicative character, define the Gauss sum

$$G_\psi(\chi) = \sum_{x \in k^\times} \chi(x) \psi(x).$$

Prove that $G_\psi(\chi)G_\psi(\overline{\chi}) = q$, where $\overline{\chi}$ is the character for which $\chi(x)$ is the complex conjugate of $\chi(x)$.

(2) Fix a choice of $\chi$ as above. For $P(T) = T^n + P_{n-1}T^{n-1} + \cdots + P_0 \in k[T]$ a monic polynomial, define

$$\lambda(P) = \chi(P_0)\psi(P_{n-1}).$$

(In particular, $\lambda(1) = 1$.) Show that

$$\sum_{P \in k[T]} \lambda(P) U^{\deg(P)} = \prod_{Q \in k[T]} (1 - \lambda(Q) U^{\deg(Q)})^{-1}. $$

(3) Show that for $n$ a nonnegative integer,

$$\sum_{P \in k[T]} \lambda(P) U^{\deg(P)} = \begin{cases} 
1 & \text{if } n = 0 \\
G_\psi(\chi) U & \text{if } n = 1 \\
0 & \text{if } n > 1.
\end{cases}$$

(4) With notation as in the previous problem, let $k'$ be an extension of $k$ of degree $v$. Let $\psi': k' \to \mathbb{C}^\times$ be the additive character given by $\psi \circ \text{Trace}_{k'/k}$. Given $\chi$, let $\chi'$ be the multiplicative character given by $\chi \circ \text{Norm}_{k'/k}$. For $P' \in k'[T]$ monic, define $\lambda'$ by analogy with $\lambda$.

For $P \in k[T]$ monic, let $P'$ run over the irreducible factors of $P$ in $k'[T]$. Prove that

$$\prod_{P'} (1 - \lambda'(P') U^{v \deg(P')}) = \prod_{\rho=0}^{v-1} (1 - \lambda(P)(e^{2\pi i \rho/v})U^{\deg(P)}).$$

(Hint: let $-\xi$ be a root of one of the factors $P'$, and consider the field extensions $k(\xi)/k$ and $k'(\xi)/k'$.)

(5) Using all of the above, deduce the Davenport-Hasse relation

$$-G_{\psi'}(\chi') = (-G_\psi(\chi))^v.$$