(1) Define the rings
\[ R = \mathbb{Z}[x_1, y_1, x_2, y_2, \ldots], \quad R' = \mathbb{Q}[x_1, y_1, x_2, y_2, \ldots], \quad F = \text{Frac}(R) = \text{Frac}(R'). \]
Define the power series \( x = 1 + x_1 T + x_2 T^2 + \cdots, y = 1 + y_1 T + y_2 T^2 + \cdots, \) and
\[ f = 1 / \exp(\log(1/x) \star \log(1/y)) \in R'[T] \]
where \( \star \) denotes the Hadamard product:
\[(a_1 T + a_2 T^2 + \cdots) \star (b_1 T + b_2 T^2 + \cdots) = a_1 b_1 T + a_2 b_2 T^2 + \cdots \]
(a) Let \( V_1, V_2 \) be two finite-dimensional vector spaces over \( F \) equipped with endomorphisms \( \varphi_1, \varphi_2 \) satisfying, for some positive integer \( n \),
\[ \det(1 - \varphi_1 T, V_1)^{-1} \equiv 1 + x_1 T + \cdots + x_n T^n \quad (\mod T^{n+1} F[T]), \]
\[ \det(1 - \varphi_2 T, V_2)^{-1} \equiv 1 + y_1 T + \cdots + y_n T^n \quad (\mod T^{n+1} F[T]), \]
Prove that
\[ \det(1 - (\varphi_1 \otimes \varphi_2) T, V_1 \otimes_F V_2)^{-1} \equiv f \quad (\mod T^{n+1} F[T]). \]
(Hint: pass to an algebraic closure of \( F \) and write everything in terms of eigenvalues. Remember that \( f \) is determined mod \( T^{n+1} F[T] \) by \( x_1, \ldots, x_n, y_1, \ldots, y_n. \) )
(b) Deduce that \( f \in R[T] \).
(2) Using the previous exercise, prove that there is a unique functor \( \Lambda \) from rings to rings with the following properties.
(a) The underlying functor from rings to additive groups takes \( R \) to \( \Lambda(R) = 1 + TR[T] \) with the usual series multiplication.
(b) For any ring \( R \), the multiplication map \( \star \) on \( \Lambda(R) \) satisfies
\[(1 - aT)^{-1} \star (1 - bT)^{-1} = (1 - abT)^{-1} \quad (a, b \in R). \]
The ring \( \Lambda(R) \) is (a form of) the ring of big Witt vectors with coefficients in \( R \).
(3) Let \( X_1, X_2 \) be two varieties over \( \mathbb{F}_q \). Prove that in \( \Lambda(\mathbb{Z}) \), we have
\[ Z(X_1 \times_{\mathbb{F}_q} X_2, T) = Z(X_1, T) \ast Z(X_2, T). \]
(4) Let \( K \) be a field of characteristic 0. Let \( P(x) \in K[x] \) be a monic polynomial of degree \( 2g + 1 \) with no repeated roots.
(a) Let \( X \) be the affine scheme \( \text{Spec} K[x, y]/(y^2 - P(x)) \). Prove that \( \Omega^1_{X/K} \) is freely generated by \( dx/y \). (Hint: it suffices to check that \( dx/y \) is a nowhere vanishing section of \( \Omega^1_{X/K} \). Treat the points where \( y = 0 \) and \( y \neq 0 \) separately.)
(b) Prove that \( H_{\text{dR}}^1(X) \) admits the basis
\[ x^i \frac{dx}{y} \quad (i = 0, \ldots, 2g - 1). \]
(Hint: for each integer \( d \geq 2g \), write down a relation of the form \( Q(x) dx/y \) with \( \deg(Q) = d. \))
(c) Let $Y$ be the affine scheme $\text{Spec } K[x, y, z]/(y^2 - P(x), yz - 1)$. Prove that $H_{dR}^1(Y)$ admits the basis

$$x^i \frac{dx}{y}, \quad (i = 0, \ldots, 2g - 1); \quad x^i \frac{dx}{y^2}, \quad (i = 0, \ldots, 2g).$$

(5) Let $p > 2$ be a prime. Let $\overline{P} \in \mathbb{F}_p[x]$ be a monic polynomial of degree $2g + 1$ with no repeated roots.

(a) Put $\overline{X} = \text{Spec } \mathbb{F}_p[x, y]/(y^2 - \overline{P}(x))$. Prove that $H_{MW}^1(\overline{X})$ admits the basis

$$x^i \frac{dx}{y}, \quad (i = 0, \ldots, 2g - 1).$$

(b) Put $\overline{Y} = \text{Spec } \mathbb{F}_p[x, y, z]/(y^2 - \overline{P}(x), yz - 1)$. Prove that $H_{MW}^1(\overline{Y})$ admits the basis

$$x^i \frac{dx}{y}, \quad (i = 0, \ldots, 2g - 1); \quad x^i \frac{dx}{y^2}, \quad (i = 0, \ldots, 2g).$$