Extra Problems 2
The following includes textbook problems and problems from past Math 103A courses to give you additional practice for the exam. However, you should still review all past homework problems, lecture notes, discussion notes, and sections 8 through 11 of the textbook. These sections naturally rely on previous content, so it is worth reviewing important definitions or results from the previous sections as well. Complex numbers and linear algebra will not appear in the exam.

1. Find the maximum possible order of an element \( \sigma \in S_7 \), and give an example of such a \( \sigma \).

2. Let \( \sigma = (a_1\ a_2\ \ldots\ a_k) \in S_n \). For any \( \tau \in S_n \), prove that \( \tau \sigma \tau^{-1} = (\tau(a_1)\ \tau(a_2)\ \ldots\ \tau(a_k)) \).

3. (§8, #47) Show that if \( n \geq 3 \), then the only element \( \sigma \) of \( S_n \) satisfying \( \sigma \gamma = \gamma \sigma \) for all \( \gamma \in S_n \) is \( \sigma = \iota \).

4. (a) Let \( n \geq 3 \). Describe an algorithm for writing any \( \sigma \in A_n \) as a product of 3-cycles. 
   \textit{Hint: For distinct } a, b, c, d, \textit{ (a b)(b c) = (a b c) and (a b)(c d) = (a b c)(b c d).}
   
   (b) Is the statement true for odd permutations? Explain.

5. Let \( G = \langle a \rangle \) be a cyclic group of order 15. Find all of the left cosets of \( H = \langle a^5 \rangle \) in \( G \).

6. (§10, #38) Prove Theorem 10.14. \textit{[Hint: Let } \{a_i H_i \mid i = 1, \ldots, r\} \text{ be the collection of distinct left cosets of } H \text{ in } G \text{ and } \{b_j K_j \mid j = 1, \ldots, s\} \text{ be the collection of distinct left cosets of } K \text{ in } H \text{. Show that}

   \[ \{ (a_i b_j) K \mid i = 1, \ldots, r; j = 1, \ldots, s \} \]

   \textit{is the collection of distinct left cosets of } K \text{ in } G \text{.}]

7. Let \( G \) be a finite group with an odd number of elements.
   
   (a) Prove that \( x^2 = e \) has a unique solution in \( G \).
   
   (b) If \( G \) is abelian, show that the product of all of the elements of \( G \) is \( e \).

8. (a) (§11, #18) Are the groups \( \mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24} \) and \( \mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40} \) isomorphic? Why or why not?
   
   (b) (§11, #20) Are the groups \( \mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{15} \) and \( \mathbb{Z}_3 \times \mathbb{Z}_{36} \times \mathbb{Z}_{10} \) isomorphic? Why or why not?

9. (a) Let \( G_1 \) and \( G_2 \) be groups, and let \( H_1 \leq G_1 \) and \( H_2 \leq G_2 \). Prove that \( H_1 \times H_2 \leq G_1 \times G_2 \).
   
   (b) Find an example of groups \( G_1 \) and \( G_2 \) and a subgroup \( H \) of \( G_1 \times G_2 \) such that \( H \) is not a direct product of a subgroup of \( G_1 \) and a subgroup of \( G_2 \).