Extra Problems 2 Solutions

1. Find the maximum possible order of an element \( \sigma \in S_7 \), and give an example of such a \( \sigma \).

The following are all partitions of 7:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>4 + 2 + 1</td>
<td>3 + 1 + 1 + 1 + 1</td>
</tr>
<tr>
<td>6 + 1</td>
<td>4 + 1 + 1 + 1</td>
<td>2 + 2 + 2 + 1</td>
</tr>
<tr>
<td>5 + 2</td>
<td>3 + 3 + 1</td>
<td>2 + 2 + 1 + 1 + 1</td>
</tr>
<tr>
<td>5 + 1 + 1</td>
<td>3 + 2 + 2</td>
<td>2 + 1 + 1 + 1 + 1 + 1</td>
</tr>
<tr>
<td>4 + 3</td>
<td>3 + 2 + 1 + 1</td>
<td>1 + 1 + 1 + 1 + 1 + 1 + 1</td>
</tr>
</tbody>
</table>

Since these partitions correspond to the possible cycle types of elements in \( S_7 \), the maximum possible order of an element in \( S_7 \) is the maximum least common multiple of the summands of a partition, which is \( \text{lcm}(4, 3) = 12 \). For example, \( \sigma = (1 \ 2 \ 3 \ 4)(5 \ 6 \ 7) \in S_7 \) has order 12.

2. Let \( \sigma = (a_1 \ a_2 \ldots \ a_k) \in S_n \). For any \( \tau \in S_n \), prove that \( \tau \sigma \tau^{-1} = (\tau(a_1) \ \tau(a_2) \ldots \ \tau(a_k)) \).

Proof. Let \( \sigma = (a_1 \ a_2 \ldots \ a_k) \in S_n \) and \( \tau \in S_n \). For \( i = 1, 2, \ldots, k - 1, \)

\[ \tau \sigma \tau^{-1}(\tau(a_i)) = \tau \sigma(a_i) = \tau(a_{i+1}) . \]

Moreover,

\[ \tau \sigma \tau^{-1}(\tau(a_k)) = \tau \sigma(a_k) = \tau(a_1) , \]

so \( \{\tau(a_1), \tau(a_2), \ldots, \tau(a_k)\} \) is an orbit of \( \tau \sigma \tau^{-1} \).

If \( y \in \{1, 2, \ldots, n\} - \{\tau(a_1), \tau(a_2), \ldots, \tau(a_k)\} \), then \( y = \tau(x) \) for some \( x \not\in \{a_1, a_2, \ldots, a_k\} \) since \( \tau \) is bijective. Then

\[ \tau \sigma \tau^{-1}(y) = \tau \sigma \tau^{-1}(\tau(x)) = \tau \sigma(x) = \tau(x) = y \]

since \( \sigma \) fixes \( x \). Therefore, \( \tau \sigma \tau^{-1} \) is the cycle \( (\tau(a_1) \ \tau(a_2) \ldots \ \tau(a_k)) \). \( \Box \)

3. (§8, #47) Show that if \( n \geq 3 \), then the only element \( \sigma \) of \( S_n \) satisfying \( \sigma \gamma = \gamma \sigma \) for all \( \gamma \in S_n \) is \( \sigma = \iota \).

Proof. Let \( n \geq 3 \) and \( \sigma \in S_n \). Suppose that \( \sigma \gamma = \gamma \sigma \) for all \( \gamma \in S_n \). If \( \sigma \neq \iota \), then \( \sigma(a) = b \) for some \( a \neq b \) in \( \{1, 2, \ldots, n\} \). Take some \( \gamma \in S_n \) such that \( \gamma(a) = a \) and
\[ \gamma(b) = c \text{ for some } c \not\in \{a, b\} \text{ (which exists since } n \geq 3) \text{. Then} \]
\[ \sigma\gamma(a) = \sigma(a) = b \]
\[ \gamma\sigma(a) = \gamma(b) = c, \]
so \( \sigma\gamma \neq \gamma\sigma \), which is a contradiction. Therefore, \( \sigma = \iota \). \qed

4. (a) Let \( n \geq 3 \). Describe an algorithm for writing any \( \sigma \in A_n \) as a product of 3-cycles. 
   \textit{Hint:} For distinct \( a, b, c, d \), \((a b)(b c) = (a b c) \) and \((a b)(c d) = (a b c)(b c d)\).
   Since any \( \sigma \in A_n \) can be written as the product of an even number of transpositions, we can write \( \sigma \) as \( \tau_1 \tau_2 \tau_3 \cdots \tau_k \) for some transpositions \( \tau_i \). Take any pair \( \tau_i \tau'_i \) in the decomposition. If \( \tau_i = \tau'_i \), then their product is \( \iota \). Otherwise, \( \tau_i \) and \( \tau'_i \) share exactly 1 element or are disjoint, so using the hint, we can replace the pair with a 3-cycle.

(b) Is the statement true for odd permutations? Explain.
   No. Since any cycle of length 3 is even, any product of 3-cycles must also be even.

5. Let \( G = \langle a \rangle \) be a cyclic group of order 15. Find all of the left cosets of \( H = \langle a^5 \rangle \) in \( G \).
   Since \( G = \langle a \rangle \) has order 15, then \( H = \langle a^5 \rangle = \{e, a^5, a^{10}\} \). The left cosets of \( H \) in \( G \) are
   \[ H = \{e, a^5, a^{10}\} \]
   \[ aH = \{a, a^6, a^{11}\} \]
   \[ a^2H = \{a^2, a^7, a^{12}\} \]
   \[ a^3H = \{a^3, a^8, a^{13}\} \]
   \[ a^4H = \{a^4, a^9, a^{14}\} \]

6. (§10, #38) Prove Theorem 10.14. \textit{[Hint:} Let \( \{a_iH \mid i = 1, \ldots, r\} \) be the collection of distinct left cosets of \( H \) in \( G \) and \( \{b_jK \mid j = 1, \ldots, s\} \) be the collection of distinct left cosets of \( K \) in \( H \). Show that \( \{(a_ib_j)K \mid i = 1, \ldots, r; j = 1, \ldots, s\} \) is the collection of distinct left cosets of \( K \) in \( G \).\]
   \textit{Proof.} Let \( A = \{a_iH \mid i = 1, \ldots, r\} \) be the collection of distinct left cosets of \( H \) in \( G \) and \( B = \{b_jK \mid j = 1, \ldots, s\} \) be the collection of distinct left cosets of \( K \) in \( H \). Consider the set \( C = \{(a_ib_j)K \mid i = 1, \ldots, r; j = 1, \ldots, s\} \). We’ll show that every element in \( G \) lives in some coset in \( C \), and that the cosets in \( C \) are pairwise disjoint. Let \( g \in G \). Then \( g \in a_iH \) for some \( i \in \{1, \ldots, r\} \), so
(a) \( g = a_i h \) for some \( h \in H \). Since \( h \in H \), then \( h \in b_j K \) for some \( j \in \{1, \ldots, s\} \), so \( h = b_j k \) for some \( k \in K \). Then \( g = a_i h = a_i b_j k \in (a_i b_j) K \). Now take any \((a_i b_j) K \) and \((a_m b_n) K \) in \( C \). If there exists some \( g \in (a_i b_j) K \cap (a_m b_n) K \), then \( g = (a_i b_j) k_1 \) and \( g = (a_m b_n) k_2 \) for some \( k_1, k_2 \in K \). Since \( b_j, b_n \in H \) and \( K \leq H \), then \( b_j k_1, b_n k_2 \in H \). Therefore, \( g = a_i (b_j k_1) \in a_i H \) and \( g = a_m (b_n k_2) \in a_m H \). Since the cosets in \( A \) are distinct and hence pairwise disjoint, \( i = m \). Notice that \( a_i^{-1} g = b_j k_1 \in b_j K \) and \( a_m^{-1} g = b_n k_2 \in b_n K \). Since \( i = m \) and the cosets in \( B \) are pairwise disjoint, then \( j = n \). Hence, the cosets in \( C \) are pairwise disjoint, so \( C \) is the set of distinct left cosets of \( K \) in \( G \). \( \square \)

7. Let \( G \) be a finite group with an odd number of elements.

(a) Prove that \( x^2 = e \) has a unique solution in \( G \).

\[ \textbf{Proof.} \] Let \( G \) be a finite group with an odd number of elements. For any \( a \in G \), if \( a^2 = e \), then the order of \( a \) divides 2, so the order of \( a \) is 1 or 2. By Lagrange’s Theorem, the order of \( a \) must divide the order of \( G \), so the order of \( a \) cannot be 2 as the order of \( G \) is odd. Therefore, the order of \( a \) is 1, so \( a = e \). Therefore, the only solution to \( x^2 = e \) in \( G \) is \( x = e \). \( \square \)

(b) If \( G \) is abelian, show that the product of all of the elements of \( G \) is \( e \).

\[ \textbf{Proof.} \] Let \( |G| = 2k + 1 \) for some nonnegative integer \( k \). If \( k = 0 \), then \( |G| = 1 \), so \( G \) contains only \( e \), so we are done. Otherwise, let \( k \in \mathbb{Z}^+ \) and \( a_1, a_2, \ldots, a_{2k+1} \) be the elements of \( G \). By part (a), the only element of order 2 is \( e \), so for any nonidentity element \( b \in G \), \( b^{-1} \neq b \). We can then list the elements of \( G \) as \( e, b_1, b_1^{-1}, b_2, b_2^{-1}, \ldots, b_k, b_k^{-1} \). Therefore, since \( G \) is abelian,

\[
a_1 a_2 \cdots a_{2k+1} = e b_1 b_1^{-1} b_2 b_2^{-1} \cdots b_k b_k^{-1} = e e e \cdots e = e.
\]

\( \square \)

8. (a) \((\S 11, \#18)\) Are the groups \( \mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24} \) and \( \mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40} \) isomorphic? Why or why not?

No. Using the fact that \( \mathbb{Z}_m \times \mathbb{Z}_n \simeq \mathbb{Z}_{mn} \) if and only if \( \gcd(m, n) = 1 \), we have

\[
\mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24} \simeq \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_8 \\
\simeq \mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_5
\]

and

\[
\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40} \simeq \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_8 \\
\simeq \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_5.
\]
Notice that $\mathbb{Z}_2 \times \mathbb{Z}_8 \ncong \mathbb{Z}_4 \times \mathbb{Z}_4$. 

(b) $(\S 11, \#20)$ Are the groups $\mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{15}$ and $\mathbb{Z}_3 \times \mathbb{Z}_{36} \times \mathbb{Z}_{10}$ isomorphic? Why or why not?

Yes. Using the fact that $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ if and only if $\gcd(m,n) = 1$, we have

\[
\mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{15} \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \\
\cong \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \\
\cong \mathbb{Z}_3 \times \mathbb{Z}_{36} \times \mathbb{Z}_{10}.
\]

9. (a) Let $G_1$ and $G_2$ be groups, and let $H_1 \leq G_1$ and $H_2 \leq G_2$. Prove that $H_1 \times H_2 \leq G_1 \times G_2$.

Let $G_1, G_2$ be groups and $H_1 \leq G_1, H_2 \leq G_2$.

- Let $(a_1, a_2), (b_1, b_2) \in H_1 \times H_2$. Since $H_1$ and $H_2$ are closed under their respective binary operations, $a_1 b_1 \in H_1$ and $a_2 b_2 \in H_2$. Then $(a_1, a_2)(b_1, b_2) = (a_1 b_1, a_2 b_2) \in H_1 \times H_2$.
- Since $H_1$ contains the identity element $e_1$ of $G_1$ and $H_2$ contains the identity element $e_2$ of $G_2$, $(e_1, e_2) \in H_1 \times H_2$.
- Let $(a_1, b_1) \in H_1 \times H_2$. Since $a_1 \in H_1, a_1^{-1} \in H_1$, and since $a_2 \in H_2, a_2^{-1} \in H_2$. Thus, $(a_1^{-1}, b_1^{-1}) \in H_1 \times H_2$.

Therefore, $H_1 \times H_2 \leq G_1 \times G_2$.

(b) Find an example of groups $G_1$ and $G_2$ and a subgroup $H$ of $G_1 \times G_2$ such that $H$ is not a direct product of a subgroup of $G_1$ and a subgroup of $G_2$.

Take $G_1 = G_2 = \mathbb{Z}$ and the subgroup $H = \langle (1,1) \rangle = \{(d,d) \mid d \in \mathbb{Z}\}$ of $\mathbb{Z} \times \mathbb{Z}$. Say that $H = H_1 \times H_2$ for some subgroups $H_1, H_2$ of $\mathbb{Z}$. Since $(1, 1) \in H = H_1 \times H_2$, then $1 \in H_1$. Since $H_2$ is a subgroup of $\mathbb{Z}$, $0 \in H_2$. Therefore, $(1, 0) \in H_1 \times H_2$, but $(1, 0) \neq (d, d)$ for any $d \in \mathbb{Z}$. Therefore, $H$ is not a direct product of subgroups of $\mathbb{Z}$. 