Extra Problems 3
The following includes textbook problems and problems from past Math 103A courses to
give you additional practice for the exam. Since the exam is cumulative, you should still
review all past homework problems, lecture notes, discussion notes, and sections 0–6, 8–11,
13–16 of the textbook. Complex numbers and linear algebra will not appear on the exam.

1. Let $H$ be a subgroup of $\mathbb{Z}$ (under addition). Show that $K = \{3^a \mid a \in H\}$ is a subgroup of $\mathbb{R}^*$ (the group of nonzero real numbers under multiplication).

2. Consider the group $G = \mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_2$.
   (a) Find the order of $(3, 4, 0)$ in $\mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_2$.
   (b) Let $H = \langle (3, 4, 0) \rangle$. List the elements of $H$.
   (c) Why does $G/H$ form a group?
   (d) The following left cosets of $H$ in $G$ are distinct. Find the order of each coset.

   $$(0, 1, 0) + H, \quad (0, 0, 1) + H, \quad (0, 1, 1) + H$$
   (e) What group is $G/H$ isomorphic to?

3. (a) Prove that if $G$ is cyclic, then for any normal subgroup $H$ of $G$, $G/H$ is also cyclic.
    (b) Find a generator of $\mathbb{Z}_{100}/\langle 14 \rangle$.

4. Let $G$ be a group, $H$ be a subgroup of $G$, and $N$ be a normal subgroup of $G$. Show
   that
   $$HN = \{hn \mid h \in H, n \in N\}$$
   is a subgroup of $G$.

5. Let $\phi : G \to G'$ be a group homomorphism and let $N$ be a normal subgroup of $G$. Show
   that $\phi[N]$ is a normal subgroup of $\phi[G]$.

6. Let $\phi : G \to G'$ be a group homomorphism and let $N'$ be a normal subgroup of $G'$. Show
   that $\phi^{-1}[N']$ is a normal subgroup of $G$.

7. Show that the map $*$ is an action of the group $G$ on the set $X$.
   (a) Let $G$ be any group and take any subgroup $H$ of $G$. Let $X = G/H$. Define
       $$g * (aH) = (ga)H$$
       for all $g \in G$ and $aH \in G/H$.
   (b) Let $G = \mathbb{R}$ (under addition) and let $X$ be the set of real-valued functions defined
       on $\mathbb{R}$. Define $r * f(x) = f(x + r)$ for all $r \in \mathbb{R}$ and $f \in X$.

8. Let $S_n$ act on the set of ordered pairs $X = \{(i, j) \mid i = 1, 2, \ldots, n; j = 1, 2, \ldots, n\}$ by
   $$\sigma(i, j) = (\sigma(i), \sigma(j))$$
   for all $\sigma \in S_n$ and $(i, j) \in X$.
   (a) Show that this is a group action of $S_n$ on $X$.
   (b) Find the orbits of $S_n$ on $X$.
   (c) For each orbit $O$ of $S_n$ acting on $X$, pick some $(i, j) \in O$ and find the stabilizer
       of $(i, j)$ in $S_n$. 