Homework 1 Solutions

1. Let $A, B \in M_n(\mathbb{R})$. We say that $B$ is similar to $A$ if there exists an invertible matrix $P$ such that $B = P^{-1}AP$. Prove that similarity is an equivalence relation on $M_n(\mathbb{R})$.

Proof. Let $A, B, C \in M_n(\mathbb{R})$.

- Since $I^{-1}AI = IAI = (IA)I = AI = A$, $A$ is similar to $A$.
- Suppose that $A$ is similar to $B$. Then there exists an invertible matrix $P$ such that $A = P^{-1}BP$. Then

\[
A = P^{-1}BP \\
\Rightarrow PA = BP \\
\Rightarrow PAP^{-1} = B \\
\Rightarrow (P^{-1})^{-1}AP^{-1} = B.
\]

Since $P^{-1}$ is invertible, $B$ is similar to $A$.

- Suppose that $A$ is similar to $B$ and $B$ is similar to $C$. Then there exist invertible matrices $P$ and $Q$ such that $A = P^{-1}BP$ and $B = Q^{-1}CQ$. This implies that

\[
A = P^{-1}BP \\
\Rightarrow A = P^{-1}Q^{-1}CQP \\
\Rightarrow A = (QP)^{-1}CQP.
\]

Since $Q$ and $P$ are both invertible, so is $QP$. Therefore, $A$ is similar to $C$.

We’ve shown that the relation is reflexive, symmetric, and transitive, so similarity is an equivalence relation on $M_n(\mathbb{R})$. □

2. Find all solutions in $\mathbb{C}$ of the given equation.

(a) $z^3 = 8i$

Let $z = |z|(\cos \theta + i \sin \theta)$. If $z^3 = 8i$, then $|z|^3(\cos(3\theta) + i \sin(3\theta)) = 8(0 + i)$. This implies that

\[
|z|^3 = 8 \Rightarrow |z| = 2 \\
\cos(3\theta) = 0 \text{ and } \sin(3\theta) = 1 \Rightarrow 3\theta = \frac{\pi}{2} + 2\pi k \text{ for } k \in \mathbb{Z} \\
\Rightarrow \theta = \frac{\pi}{6} + \frac{2\pi}{3} k \text{ for } k \in \mathbb{Z}
\]
If $0 \leq \theta < 2\pi$, then $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$. Therefore, we have three solutions:

$$z_1 = 2\left(\cos \frac{\pi}{6} + i\sin \frac{\pi}{6}\right) = \sqrt{3} + i$$

$$z_2 = 2\left(\cos \frac{5\pi}{6} + i\sin \frac{5\pi}{6}\right) = -\sqrt{3} + i$$

$$z_3 = 2\left(\cos \frac{3\pi}{2} + i\sin \frac{3\pi}{2}\right) = -2i$$

(b) $z^6 = -27$

Let $z = |z|(\cos \theta + i\sin \theta)$. If $z^6 = -27$, then $|z|^6(\cos (6\theta) + i\sin (6\theta)) = 27(-1 + 0i)$. This implies that

$$|z|^6 = 27 \Rightarrow |z| = \sqrt{3}$$

$\cos (6\theta) = -1$ and $\sin (6\theta) = 0 \Rightarrow 6\theta = \pi + 2\pi k$ for $k \in \mathbb{Z}$

$$\Rightarrow \theta = \frac{\pi}{6} + \frac{\pi}{3} k$$

If $0 \leq \theta < 2\pi$, then $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$. Therefore, we have three solutions:

$$z_1 = \sqrt{3}\left(\cos \frac{\pi}{6} + i\sin \frac{\pi}{6}\right) = \frac{3}{2} + \frac{\sqrt{3}}{2}i$$

$$z_2 = \sqrt{3}\left(\cos \frac{5\pi}{6} + i\sin \frac{5\pi}{6}\right) = -\frac{3}{2} + \frac{\sqrt{3}}{2}i$$

$$z_3 = \sqrt{3}\left(\cos \frac{7\pi}{6} + i\sin \frac{7\pi}{6}\right) = -\frac{3}{2} - \frac{\sqrt{3}}{2}i$$

$$z_4 = \sqrt{3}\left(\cos \frac{3\pi}{2} + i\sin \frac{3\pi}{2}\right) = -\sqrt{3}i$$

$$z_5 = \sqrt{3}\left(\cos \frac{11\pi}{6} + i\sin \frac{11\pi}{6}\right) = \frac{3}{2} - \frac{\sqrt{3}}{2}i$$

3. Find all elements of $U_4$, in other words, the 4th roots of unity.

$$U_4 = \{1, \zeta, \zeta^2, \zeta^3\} \text{ where } \zeta = e^{\frac{2\pi i}{4}} = e^{\frac{\pi i}{2}} = \cos \frac{\pi}{2} + i\sin \frac{\pi}{2} = i. \text{ Therefore, } U_4 = \{1, i, -1, -i\}.$$

4. (§1, #41) Derive Euler’s formula, $e^{i\theta} = \cos \theta + i\sin \theta$, using the series expansions of $e^x$, $\cos x$, and $\sin x$. 

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Proof.

\[ e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \]

\[ = 1 + i\theta \left( \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^8}{8!} + \cdots \right) \]

\[ = 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^6}{6!} + \cdots + i\theta + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^7}{7!} + \cdots \]

\[ = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots + i \left( \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \right) \]

\[ = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} \]

\[ = \cos \theta + i \sin \theta \]

5. Determine whether the binary operation \( \ast \) is commutative and whether it is associative. Justify your answers.

(a) the operation \( \ast \) on \( \mathbb{R} \) defined by \( a \ast b = a + b + ab \)

Proof. The operation is both commutative and associative. Let \( a, b, c \in \mathbb{R} \). Then

- \( a \ast b = a + b + ab = b + a + ba = b \ast a \), so \( \ast \) is commutative by the commutativity of addition and multiplication of \( \mathbb{R} \), and

- \[
\begin{align*}
(a \ast b) \ast c &= (a + b + ab) \ast c \\
&= a + b + ab + c + (a + b + ab)c \\
&= a + b + ab + c + ac + bc + abc \\
&= a + b + c + ab + ac + bc + abc
\end{align*}
\]

and

\[
\begin{align*}
a \ast (b \ast c) &= a \ast (b + c + bc) \\
&= a + b + c + bc + a(b + c + bc) \\
&= a + b + c + bc + ab + ac + abc \\
&= a + b + c + ab + ac + bc + abc,
\end{align*}
\]

so \( \ast \) is associative by the commutativity of addition and the distributive laws of addition and multiplication of \( \mathbb{R} \).
(b) the operation $\ast$ on $\mathbb{Q} - \{0\}$ defined by $a \ast b = \frac{a}{b}$

Proof. The operation is neither commutative nor associative.
- Since $1 \ast 2 = \frac{1}{2} \neq 2 = 2 \ast 1$, $\ast$ is not commutative.
- Since $(1 \ast 2) \ast 3 = \frac{1}{2} \ast 3 = \frac{1}{6}$
  but $1 \ast (2 \ast 3) = 1 \ast \frac{2}{3} = \frac{1}{2} = \frac{3}{2}$,
  $\ast$ is not associative.

6. (§2, #23) Let $H$ be the subset of $M_2(\mathbb{R})$ consisting of matrices of the form

\[
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix}
\]

for $a, b \in \mathbb{R}$. Is $H$ closed under

(a) matrix addition?

Proof. Yes. Let

\[
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix}, \begin{bmatrix}
c & -d \\
d & c
\end{bmatrix} \in H
\]

Then

\[
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix} + \begin{bmatrix}
c & -d \\
d & c
\end{bmatrix} = \begin{bmatrix}
a + c & -b - d \\
b + d & a + c
\end{bmatrix}.
\]

Since $a + c, b + d \in \mathbb{R}$, $\begin{bmatrix}
a + c & -b - d \\
b + d & a + c
\end{bmatrix} \in H$, so $H$ is closed under addition.

(b) matrix multiplication?

Proof. Yes. Let

\[
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix}, \begin{bmatrix}
c & -d \\
d & c
\end{bmatrix} \in H
\]

Then

\[
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix} \begin{bmatrix}
c & -d \\
d & c
\end{bmatrix} = \begin{bmatrix}
ac - bd & -ad - bc \\
ad + bc & ac - bd
\end{bmatrix} = \begin{bmatrix}
ac - bd & -(ad + bc) \\
ad + bc & ac - bd
\end{bmatrix}.
\]

Since $ac - bd, ad + bc \in \mathbb{R}$, $\begin{bmatrix}
ac - bd & -(ad + bc) \\
ad + bc & ac - bd
\end{bmatrix} \in H$, so $H$ is closed under multiplication.

Justify your answers.

7. Prove or give a counterexample.

(a) (§2, #27) Every binary operation on a set consisting of a single element is both commutative and associative.
Proof. True. Let \(*\) be a binary operation on the set \(\{a\}\). Then \(a \ast a = a\).

Since \(a \ast a = a \ast a\), \(\ast\) is commutative. Since \((a \ast a) \ast a = a \ast a = a \ast (a \ast a)\), \(\ast\) is associative.

\[\]

(b) (§2, #28) Every commutative binary operation on a set having just two elements is associative.

Proof. False. Consider the binary operation \(\ast\) on the set \(\{a, b\}\) defined as follows:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
</tr>
</tbody>
</table>

Since \(a \ast b = a = b \ast a\), \(\ast\) is commutative. However, \((a \ast a) \ast b = b \ast b = a\) but \(a \ast (a \ast b) = a \ast a = b\), so \(\ast\) is not associative.

8. (§2, #36) Suppose that \(\ast\) is an associative binary operation on a set \(S\). Let

\[H = \{a \in S \mid a \ast x = x \ast a \text{ for all } x \in S\}\]

Show that \(H\) is closed under \(\ast\). (We think of \(H\) as consisting of all elements of \(S\) that commute with every element in \(S\).)

Proof. Let \(a, b \in H\). If \(x \in S\), then

\[(a \ast b) \ast x = a \ast (b \ast x)\]
\[= a \ast (x \ast b)\]
\[= (a \ast x) \ast b\]
\[= (x \ast a) \ast b\]
\[= x \ast (a \ast b)\]

Since \(a \ast b\) commutes with all elements of \(S\), \(a \ast b \in H\), so \(H\) is closed under \(\ast\).

9. (§2, #37) Suppose that \(\ast\) is an associative and commutative binary operation on a set \(S\). Show that

\[H = \{a \in S \mid a \ast a = a\}\]

is closed under \(\ast\). (The elements of \(H\) are idempotents of the binary operation \(\ast\).)
Proof. Let $a, b \in H$. Then

\[
(a \ast b) \ast (a \ast b) = (a \ast b) \ast (b \ast a) \quad \text{since } \ast \text{ is commutative}
\]
\[
= a \ast (b \ast (b \ast a)) \quad \text{since } \ast \text{ is associative}
\]
\[
= a \ast ((b \ast b) \ast a) \quad \text{since } \ast \text{ is associative}
\]
\[
= a \ast (b \ast a) \quad \text{since } b \in H
\]
\[
= a \ast (a \ast b) \quad \text{since } \ast \text{ is commutative}
\]
\[
= (a \ast a) \ast b \quad \text{since } \ast \text{ is associative}
\]
\[
= a \ast b \quad \text{since } a \in H.
\]

Since $(a \ast b) \ast (a \ast b) = a \ast b$, $a \ast b \in H$, so $H$ is closed under $\ast$. \qed