1. Consider $\sigma, \tau \in S_6$ where $\sigma$ is the permutation

\[
\begin{align*}
1 &\mapsto 4 \\
2 &\mapsto 2 \\
3 &\mapsto 6 \\
4 &\mapsto 5 \\
5 &\mapsto 1 \\
6 &\mapsto 3
\end{align*}
\]

and $\tau$ is the permutation

\[
\begin{align*}
1 &\mapsto 6 \\
2 &\mapsto 3 \\
3 &\mapsto 1 \\
4 &\mapsto 5 \\
5 &\mapsto 4 \\
6 &\mapsto 2
\end{align*}
\]

(a) Express each of the following as a product of disjoint cycles: $\sigma$, $\tau$, $\sigma^{-1}$, $\sigma\tau$.

\[
\begin{align*}
\sigma & = (1 \ 4 \ 5)(3 \ 6) \\
\tau & = (1 \ 6 \ 2 \ 3)(4 \ 5) \\
\sigma^{-1} & = (1 \ 5 \ 4)(3 \ 6) \\
\sigma\tau & = (1 \ 4 \ 5)(3 \ 6)(1 \ 6 \ 2 \ 3)(4 \ 5) = (1 \ 3 \ 4)(2 \ 6)
\end{align*}
\]

(b) Express each of the following as a product of transpositions, and determine which are even and which are odd: $\sigma$, $\tau$, $\sigma^{-1}$, $\sigma\tau$.

\[
\begin{align*}
\sigma & = (1 \ 5)(1 \ 4)(3 \ 6) \Rightarrow \text{odd}. \\
\tau & = (1 \ 3)(1 \ 2)(1 \ 6)(4 \ 5) \Rightarrow \text{even}. \\
\sigma^{-1} & = (1 \ 4)(1 \ 5)(3 \ 6) \Rightarrow \text{odd}. \\
\sigma\tau & = (1 \ 4)(1 \ 3)(2 \ 6) \Rightarrow \text{odd}.
\end{align*}
\]

2. Define the map $\text{sgn}: S_n \rightarrow \{\pm 1\}$ where $\{\pm 1\}$ is a group under multiplication and

\[
\text{sgn}(\sigma) = \begin{cases} 
1 & \text{if } \sigma \text{ is even} \\
-1 & \text{if } \sigma \text{ is odd}
\end{cases}
\]

Prove that $\text{sgn}$ satisfies the homomorphism property.
Proof. Let $\sigma, \tau \in S_n$.

- If $\sigma$ and $\tau$ are even, then both $\sigma$ and $\tau$ can be written as a product of an even number of transpositions. Since the sum of two even numbers is even, $\sigma \tau$ is also even. So $\text{sgn}(\sigma \tau) = 1 = 1 \cdot 1 = \text{sgn}(\sigma) \text{sgn}(\tau)$.

- If $\sigma$ is even and $\tau$ is odd, then $\sigma$ can be written as a product of an even number of transpositions and $\tau$ can be written as a product of an odd number of transpositions. Since the sum of an even number and an odd number is odd, $\sigma \tau$ is odd. So $\text{sgn}(\sigma \tau) = -1 = 1 \cdot (-1) = \text{sgn}(\sigma) \text{sgn}(\tau)$.

- If $\sigma$ is odd and $\tau$ is even, then $\sigma$ can be written as a product of an odd number of transpositions and $\tau$ can be written as a product of an even number of transpositions. Since the sum of an odd number and an even number is odd, $\sigma \tau$ is odd. So $\text{sgn}(\sigma \tau) = -1 = (-1) \cdot 1 = \text{sgn}(\sigma) \text{sgn}(\tau)$.

- If $\sigma$ and $\tau$ are odd, then both $\sigma$ and $\tau$ can be written as a product of an odd number of transpositions. Since the sum of two odd numbers is even, $\sigma \tau$ is even. So $\text{sgn}(\sigma \tau) = 1 = (-1) \cdot (-1) = \text{sgn}(\sigma) \text{sgn}(\tau)$.

Therefore, $\text{sgn}(\sigma \tau) = \text{sgn}(\sigma) \text{sgn}(\tau)$. \qed

3. Let $\sigma = (a_1 \ a_2 \ldots \ a_m), \tau = (b_1 \ b_2 \ldots \ b_l) \in S_n$ be disjoint cycles. Show that $\sigma \tau = \tau \sigma$.

Proof. Let $\sigma = (a_1 \ a_2 \ldots \ a_m), \tau = (b_1 \ b_2 \ldots \ b_l) \in S_n$ be disjoint cycles. Let $A = \{a_1, a_2, \ldots, a_m\}, B = \{b_1, b_2, \ldots, b_l\}$, and let $x \in \{1, 2, \ldots, n\}$. Since $A$ and $B$ are disjoint, then $x \in A, x \in B$, or $x$ is in neither $A$ nor $B$.

- If $x \in A$, then $\tau$ fixes $x$, so $\sigma \tau(x) = \sigma(x)$. Since $x \in A$, $\sigma(x)$ is also in $A$, so $\tau$ fixes $\sigma(x)$ as well. Then $\tau \sigma(x) = \sigma(x)$.

- If $x \in B$, then $\sigma$ fixes $x$, so $\tau \sigma(x) = \tau(x)$. Since $x \in B$, $\tau(x)$ is also in $B$, so $\sigma$ fixes $\tau(x)$ as well. Then $\sigma \tau(x) = \tau(x)$.

- If $x \notin A$ and $x \notin B$, then both $\sigma$ and $\tau$ fix $x$, so $\sigma \tau(x) = \sigma(x) = x = \tau(x) = \tau \sigma(x)$.

Therefore, $\sigma \tau = \tau \sigma$. \qed

4. You do not need to formally prove your answers to the following:

(a) Find the inverse of the cycle $\sigma = (a_1 \ a_2 \ldots \ a_m) \in S_n$. Express your answer in cyclic notation.

$$\sigma^{-1} = (a_m \ a_{m-1} \ldots \ a_2 \ a_1)$$

(b) Find the order of a cycle $\sigma = (a_1 \ a_2 \ldots \ a_m) \in S_n$. 
(c) Find a formula for the order of any permutation $\sigma \in S_n$ based on its disjoint cycle decomposition.

If $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$ is a product of disjoint cycles and $l_i$ is the length of the cycle $\sigma_i$ (and hence the order of $\sigma_i$), then the order of $\sigma$ is the least common multiple of $l_1, l_2, \ldots, l_k$.

5. ($\S$ 9, #27, parts (a) and (b)) Prove the following about $S_n$ if $n \geq 3$.

(a) Every permutation in $S_n$ can be written as a product of at most $n - 1$ transpositions.

Proof. Let $\sigma \in S_n$ where $n \geq 3$. Suppose that $\sigma = \sigma_1 \sigma_2 \cdots \sigma_m$, a product of disjoint cycles. Then we can express each cycle $\sigma_i$ as a product of transpositions $\tau_1 \tau_2 \cdots \tau_{k_i}$. Let $l(\sigma_i)$ denote the length of $\sigma_i$. Then the number of transpositions in this decomposition of $\sigma$ is

$$\sum_{i=1}^{m} k_i = \sum_{i=1}^{m} (l(\sigma_i) - 1)$$

$$= \sum_{i=1}^{m} l(\sigma_i) - \sum_{i=1}^{m} 1$$

$$\leq n - m$$

$$\leq n - 1.$$  

(b) Every permutation in $S_n$ that is not a cycle can be written as a product of at most $n - 2$ transpositions.

Proof. This is the same proof as in part (a), with the additional assumption that $\sigma$ is not a cycle, and therefore $m \geq 2$, so $\sum_{i=1}^{m} k_i \leq n - 2$.

6. ($\S$ 9, #29) Show that for every subgroup $H$ of $S_n$ for $n \geq 2$, either all the permutations in $H$ are even or exactly half of them are even.

Proof. Let $H$ be a subgroup of $S_n$ where $n \geq 2$. Say that $H$ contains an odd permutation $\sigma$. Let $A = \{\alpha \in H \mid \alpha$ is even\}$ and $B = \{\alpha \in H \mid \alpha$ is odd\}$. Define the map $\phi: A \to B$ by $\phi(\tau) = \sigma \tau$ for any $\tau \in A$. This is well-defined since $H$ is closed multiplication and the product of an odd permutation and an even permutation is odd. The map $\psi: B \to A$ where $\psi(\tau) = \sigma^{-1} \tau$ for all $\tau \in B$ is well-defined since $H$ is closed under multiplication and the product of $\sigma^{-1}$, an odd permutation, and $\tau$, also an odd permutation, is even. Moreover, $\psi$ is the inverse of $\phi$ since $\phi \psi(\tau) = \phi(\sigma^{-1} \tau) = \sigma(\sigma^{-1} \tau) = \tau$ for all $\tau \in B$ and $\psi \phi(\tau) = \psi(\sigma \tau) = \sigma^{-1}(\sigma \tau) = \tau$ for all $\tau \in A$. Therefore, $\phi$ is a bijection. This implies that the
number of even permutations in $H$ is the same as the number of odd permutations in $H$, so exactly half of the elements of $H$ are even. If $H$ does not contain any odd permutation, then all of its elements are even.

7. (§9, #33) Consider $S_n$ for a fixed $n \geq 2$ and let $\sigma$ be a fixed odd permutation in $S_n$. Show that every odd permutation in $S_n$ is a product of $\sigma$ and some permutation in $A_n$.

Take $S_n$ where $n \geq 2$. Fix some odd $\sigma \in S_n$. Let $\tau \in S_n$ be odd. Then $\sigma^{-1}$ is also odd, and so $\sigma^{-1}\tau$ is even. Then $\tau = \sigma\alpha$, where $\alpha = \sigma^{-1}\tau \in A_n$.

8. (§9, #34) Show that if $\sigma$ is a cycle of odd length, then $\sigma^2$ is a cycle.

$\textbf{Proof.}$ Let $\sigma = (a_1 \ a_2 \ \ldots \ a_{2k} \ a_{2k+1})$ where $k$ is a positive integer. Then

$$\sigma^2 = (a_1 \ a_2 \ \ldots \ a_{2k} \ a_{2k+1})(a_1 \ a_2 \ \ldots \ a_{2k} \ a_{2k+1})$$

$$= (a_1 \ a_3 \ \ldots \ a_{2k-1} \ a_{2k+1} \ a_2 \ a_4 \ \ldots \ a_{2k-2} \ a_{2k}),$$

so $\sigma^2$ is also a cycle.

9. (a) List all of the elements of $A_4$. Write each element as a product of disjoint cycles.

$$\iota, (1 \ 2 \ 3), (1 \ 2 \ 4), (1 \ 3 \ 2), (1 \ 3 \ 4), (1 \ 4 \ 2), (1 \ 4 \ 3), (2 \ 3 \ 4), (2 \ 4 \ 3), (1 \ 2)(3 \ 4),$$

$$ (1 \ 3)(2 \ 4), (1 \ 4)(2 \ 3)$$

(b) Find a subgroup $H$ of $A_4$ of order 4. What familiar group is $H$ isomorphic to?

Take $H = \{\iota, (1 \ 2)(3 \ 4), (1 \ 3)(2 \ 4), (1 \ 4)(2 \ 3)\}$. $H$ is isomorphic to the Klein four-group.