Homework 6 Solutions

1. Find the left cosets and the right cosets of the subgroup $H$ of $G$. Is it the case that $aH = Ha$ for all $a \in G$? Also find $(G : H)$.

   (a) $H = \langle 4 \rangle$, $G = \mathbb{Z}_{20}$

<table>
<thead>
<tr>
<th>Left cosets:</th>
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<tbody>
<tr>
<td>$H = {0, 4, 8, 12, 16}$</td>
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<tr>
<td>$1 + H = {1, 5, 9, 13, 17}$</td>
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<tr>
<td>$2 + H = {2, 6, 10, 14, 18}$</td>
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<tr>
<td>$3 + H = {3, 7, 11, 15, 19}$</td>
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   Right cosets:

| $H = \{0, 4, 8, 12, 16\}$ |         |
| $H + 1 = \{1, 5, 9, 13, 17\}$ |         |
| $H + 2 = \{2, 6, 10, 14, 18\}$ |         |
| $H + 3 = \{3, 7, 11, 15, 19\}$ |         |

Yes, $a + H = H + a$ for all $a \in \mathbb{Z}_{20}$. $(G : H) = 4$

(b) $H = \langle (1 2 3) \rangle$, $G = A_4$

<table>
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<tbody>
<tr>
<td>$H = {\iota, (1 2 3), (1 3 2)}$</td>
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</tr>
<tr>
<td>$(1 2 4)H = {(1 2 4), (1 4)(2 3), (1 3 4)}$</td>
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</tr>
<tr>
<td>$(1 4 2)H = {(1 4 2), (2 3 4), (1 3)(2 4)}$</td>
<td></td>
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<tr>
<td>$(1 4 3)H = {(1 4 3), (1 2)(3 4), (2 4 3)}$</td>
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<tr>
<td>$H = {\iota, (1 2 3), (1 3 2)}$</td>
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<tr>
<td>$H(1 2 4) = {(1 2 4), (1 3)(2 4), (2 4 3)}$</td>
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</tr>
<tr>
<td>$H(1 4 2) = {(1 4 2), (1 4 3), (1 4)(2 3)}$</td>
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<tr>
<td>$H(1 3 4) = {(1 3 4), 2 3 4), (1 2)(3 4)}$</td>
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</table>
No, for example, \((1 \ 2 \ 4)H \neq H(1 \ 2 \ 4)\). \((G : H) = 4\)

(c) \(H = \{\epsilon, (1 \ 2)(3 \ 4), (1 \ 3)(2 \ 4), (1 \ 4)(2 \ 3)\}, G = A_4\)

### Left cosets:

\[
H = \{\epsilon, (1 \ 2)(3 \ 4), (1 \ 3)(2 \ 4), (1 \ 4)(2 \ 3)\} \\
(1 \ 2 \ 3)H = \{(1 \ 2 \ 3), (1 \ 3 \ 4), (2 \ 4 \ 3), (1 \ 4 \ 2)\} \\
(1 \ 2 \ 4)H = \{(1 \ 2 \ 4), (1 \ 4 \ 3), (1 \ 3 \ 2), (2 \ 3 \ 4)\}
\]

### Right cosets:

\[
H = \{\epsilon, (1 \ 2)(3 \ 4), (1 \ 3)(2 \ 4), (1 \ 4)(2 \ 3)\} \\
H(1 \ 2 \ 3) = \{(1 \ 2 \ 3), (2 \ 4 \ 3), (1 \ 4 \ 2), (1 \ 3 \ 4)\} \\
H(1 \ 2 \ 4) = \{(1 \ 2 \ 4), (2 \ 3 \ 4), (1 \ 4 \ 3), (1 \ 3 \ 2)\}
\]

Yes, \(\sigma H = H\sigma\) for all \(\sigma \in A_4\). \((G : H) = 3\)

2. Find the index of \(\langle (1 \ 3 \ 6)(2 \ 4) \rangle\) in \(S_6\).

\[
(S_6 : \langle (1 \ 3 \ 6)(2 \ 4) \rangle) = \frac{|S_6|}{|\langle (1 \ 3 \ 6)(2 \ 4) \rangle|} = \frac{6!}{6} = \frac{6!}{6 \cdot 2 \cdot 3} = \frac{6!}{6 \cdot 6} = 5! = 120
\]

3. (§10, #28) Let \(H\) be a subgroup of a group \(G\) such that \(g^{-1}hg \in H\) for all \(g \in G\) and all \(h \in H\). Show that every left coset \(gH\) is the same as the right coset \(Hg\).

**Proof.** Let \(H\) be a subgroup of a group \(G\) such that \(g^{-1}hg \in H\) for all \(g \in G\) and all \(h \in H\). Let \(g\) be any element in \(G\).

\(gH \subseteq Hg\): If \(x \in gH\), then \(x = gh\) for some \(h \in H\). By our assumption, since \(g^{-1} \in G\), \((g^{-1})^{-1}hg^{-1} \in H\). Therefore,

\[
x = gh = (g^{-1})^{-1} h = (g^{-1})^{-1} hg^{-1} g = ((g^{-1})^{-1}hg^{-1}) g \in Hg.
\]

\(Hg \subseteq gH\): If \(x \in Hg\), then \(x = hg\) for some \(h \in H\). By our assumption, since \(g \in G\), \(g^{-1}hg \in H\). Therefore,

\[
x = hg = gg^{-1}hg = g(g^{-1}hg) \in gH.
\]

Since \(gH \subseteq Hg\) and \(Hg \subseteq gH\), \(gH = Hg\) for all \(g \in G\). \(\Box\)

4. (§10, #29) Let \(H\) be a subgroup of a group \(G\). Prove that if the partition of \(G\) into left cosets of \(H\) is the same as the partition into right cosets of \(H\), then \(g^{-1}hg \in H\) for all \(g \in G\) and all \(h \in H\). (Note that this is the converse of exercise 28.)
Proof. Let $H$ be a subgroup of a group $G$. Suppose that $gH = Hg$ for all $g \in G$. Let $g \in G$ and $h \in H$. Since $hg \in Hg$ and $gH = Hg$, then $hg \in gH$, so $hg = gh'$ for some $h' \in H$. Then

$$g^{-1}hg = g^{-1}gh' = h' \in H.$$ 

5. (§10, #34) Let $G$ be a group of order $pq$, where $p$ and $q$ are prime numbers. Show that every proper subgroup of $G$ is cyclic.

Proof. Let $G$ be a group of order $pq$ where $p$ and $q$ are prime numbers. Let $H$ be a proper subgroup of $G$. By Lagrange’s Theorem, $|H|$ divides $pq$, so $|H| = 1, p, q$, or $pq$. Since $H$ is proper, $|H| \neq pq$. If $|H| = 1$, then $H$ is the trivial subgroup of $G$ and hence is cyclic. If $|H| = p$ or $|H| = q$, then $H$ is a group of prime order and hence is cyclic. Therefore, any proper subgroup of $G$ is cyclic.

6. (§10, #39) Show that if $H$ is a subgroup of index 2 in a finite group $G$, then every left coset of $H$ is also a right coset of $H$.

Proof. Let $G$ be a group and $H$ be a subgroup of index 2 in $G$. Then there are exactly two left cosets, $H$ and $aH$, and exactly two right cosets, $H$ and $Ha$, for some $a \in G - H$. Since the left cosets of $H$ in $G$ form a partition of $G$, $aH = G - H$. Similarly, since the right cosets of $H$ in $G$ form a partition of $G$, $Ha = G - H$. Therefore, $aH = Ha$, so every left coset of $H$ is also a right coset of $H$.

7. (§10, #40) Show that if a group $G$ with identity $e$ has finite order $n$, then $a^n = e$ for all $a \in G$.

Proof. Let $G$ be a group of order $n$, and let $e$ be the identity element of $G$. Let $a \in G$, and let $k$ be the order of $a$. By Lagrange’s Theorem, $k$ divides $n$, so $kd = n$ for some $d \in \mathbb{Z}$. Then $a^n = a^{kd} = (a^k)^d = e^d = e$.

8. (§11, #1) List the elements of $\mathbb{Z}_2 \times \mathbb{Z}_4$. Find the order of each of the elements. Is this group cyclic?

- $(0, 0)$, order 1
- $(0, 1)$, order 4
- $(0, 2)$, order 2
- $(0, 3)$, order 4
- $(1, 0)$, order 2
- $(1, 1)$, order 4
- $(1, 2)$, order 2
- $(1, 3)$, order 4
$\mathbb{Z}_2 \times \mathbb{Z}_4$ is not cyclic since $\gcd(2, 4) = 2 \neq 1$. (Notice that there is no element of order 8.)

9. (§11, #6) Find the order of $(3, 10, 9)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$.

\[
\text{lcm} \left( \frac{4}{\gcd(4, 3)}, \frac{12}{\gcd(12, 10)}, \frac{15}{\gcd(15, 9)} \right) = \text{lcm} \left( \frac{4}{1}, \frac{12}{2}, \frac{15}{3} \right) = \text{lcm}(4, 6, 5) = 60.
\]

10. Find all abelian groups, up to isomorphism, of order 540.

Note that $540 = 2^2 \cdot 3^3 \cdot 5$.

- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{27} \times \mathbb{Z}_5$
- $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
- $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
- $\mathbb{Z}_4 \times \mathbb{Z}_{27} \times \mathbb{Z}_5$

11. (§11, #52) Show that a finite abelian group is not cyclic if and only if it contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime $p$.

**Proof.** Let $G$ be a finite abelian group.

$(\Rightarrow)$ Suppose that $G$ is not cyclic. By the fundamental theorem of finitely generated abelian groups, $G$ is isomorphic to $\mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_n^{r_n}}$ for some primes $p_1, p_2, \ldots, p_n$. Since $G$ is not cyclic, these primes are not all distinct, so there exists some primes such that $p_i = p_j$. Without loss of generality, say that $p_1 = p_2$, so let’s call this prime $p$. Then $G$ is isomorphic to $\mathbb{Z}_{p^{r_1}} \times \mathbb{Z}_{p^{r_2}} \times \mathbb{Z}_{p^{r_3}} \times \cdots \times \mathbb{Z}_{p^{r_n}}$. Consider the subgroup $H = \langle p^{r_1-1} \rangle \times \langle p^{r_2-1} \rangle \times \{0\} \times \cdots \{0\}$ of $\mathbb{Z}_{p^{r_1}} \times \mathbb{Z}_{p^{r_2}} \times \mathbb{Z}_{p^{r_3}} \times \cdots \times \mathbb{Z}_{p^{r_n}}$. Notice that $\langle p^{r_1-1} \rangle$ is a subgroup of $\mathbb{Z}_{p^{r_1}}$ of order $\frac{p^{r_1}}{\gcd(p^{r_1}, p^{r_1-1})} = \frac{p^{r_1}}{p^{r_1-1}} = p$ and $\langle p^{r_2-1} \rangle$ is a subgroup of $\mathbb{Z}_{p^{r_2}}$ of order $\frac{p^{r_2}}{\gcd(p^{r_2}, p^{r_2-1})} = \frac{p^{r_2}}{p^{r_2-1}} = p$, so both $\langle p^{r_1-1} \rangle$ and $\langle p^{r_2-1} \rangle$ are isomorphic to $\mathbb{Z}_p$. Therefore, $H \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \{0\} \cdots \{0\} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, so $G$ contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$.

$(\Leftarrow)$ Suppose that $G$ contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. For a contradiction, say that $G$ is cyclic. Then every subgroup of $G$ is also cyclic, but this is a contradiction since $\mathbb{Z}_p \times \mathbb{Z}_p$ is not cyclic. Therefore, $G$ is not cyclic. □