Homework 7, due on Wednesday, March 6

1. Determine whether the map $\phi$ is a group homomorphism. If it is, find $\ker (\phi)$. Is $\phi$ injective?

   (a) Let $\mathbb{R}^*$ be the group of nonzero real numbers under multiplication:
   
   $\phi: \mathbb{R}^* \to \mathbb{R}^*$ where for all $x \in \mathbb{R}^*$, $\phi(x) = |x|
   
   (b) Recall that $\text{tr}(A)$ is the sum of the diagonal entries of a square matrix $A$, and $\mathbb{R}$ is a group under addition:
   
   $\phi: GL(n, \mathbb{R}) \to \mathbb{R}$ where for all $A \in GL(n, \mathbb{R})$, $\phi(A) = \text{tr}(A)$
   
   (c) $\phi: \mathbb{Z}_6 \to \mathbb{Z}_2$ where for all $x \in \mathbb{Z}_6$, $\phi(x)$ = the remainder of $x$ when divided by 2
   
   (d) $\phi: \mathbb{Z}_9 \to \mathbb{Z}_2$ where for all $x \in \mathbb{Z}_9$, $\phi(x)$ = the remainder of $x$ when divided by 2
   
   (e) Let $F$ be the additive group of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ having derivatives of all orders:
   
   $\phi: F \to F$ where for all $f \in F$, $\phi(f) = f''$
   
   (f) $\phi: S_3 \to S_3$ where for all $\sigma \in S_3$, $\phi(\sigma) = \sigma^2$

2. (#18) Compute $\ker (\phi)$ and $\phi(18)$ for $\phi: \mathbb{Z} \to \mathbb{Z}_{10}$ such that $\phi(1) = 6$.

3. Let $G$ be a group and $g \in G$. The map $i_g: G \to G$ where $i_g(x) = gxg^{-1}$ for all $x \in G$ is the inner automorphism of $G$ by $g$. Show that $i_g$ is an isomorphism.

4. (#44) Let $\phi: G \to G'$ be a group homomorphism. Show that if $|G|$ is finite, then $|\phi[G]|$ is finite and is a divisor of $|G|$.

5. (#45) Let $\phi: G \to G'$ be a group homomorphism. Show that if $|G'|$ is finite, then $|\phi[G]|$ is finite and is a divisor of $|G'|$.

6. (#47) Show that any group homomorphism $\phi: G \to G'$ where $|G|$ is a prime must either be the trivial homomorphism or a one-to-one map.

7. (#51) Let $G$ be any group and let $a$ be any element of $G$. Let $\phi: \mathbb{Z} \to G$ be defined by $\phi(n) = a^n$. Show that $\phi$ is a homomorphism. Describe the image and the possibilities for the kernel of $\phi$.

8. Let $G$ be a group. Consider the set $Z(G) = \{ z \in G \mid zg = gz \text{ for all } g \in G \}$. This set is called the center of $G$. Prove that $Z(G) \leq G$ and $Z(G)$ is normal $G$.

9. (#24) Show that $A_n$ is a normal subgroup of $S_n$ and compute $S_n/A_n$; that is, find a known group to which $S_n/A_n$ is isomorphic.

10. (#30) Let $H$ be a normal subgroup of $G$, and let $m = (G : H)$. Show that $a^m \in H$ for every $a \in G$. 