Homework 7 Solutions

1. Determine whether the map \( \phi \) is a group homomorphism. If it is, find \( \text{ker}(\phi) \). Is \( \phi \) injective?

(a) Let \( \mathbb{R}^* \) be the group of nonzero real numbers under multiplication:
\[ \phi : \mathbb{R}^* \rightarrow \mathbb{R}^* \text{ where for all } x \in \mathbb{R}^*, \phi(x) = |x| \]

Yes. For any \( x, y \in \mathbb{R}^* \), \( \phi(xy) = |xy| = |x||y| = \phi(x)\phi(y) \). If \( \phi(x) = 1 \), then \( |x| = 1 \), so \( \text{ker}(\phi) = \{±1\} \). Since \( \text{ker}(\phi) \neq \{1\} \), \( \phi \) is not injective.

(b) Recall that \( \text{tr}(A) \) is the sum of the diagonal entries of a square matrix \( A \), and \( \mathbb{R} \) is a group under addition:
\[ \phi : \text{GL}(n, \mathbb{R}) \rightarrow \mathbb{R} \text{ where for all } A \in \text{GL}(n, \mathbb{R}), \phi(A) = \text{tr}(A) \]

No. For example, \( \phi(I) = \phi(I) = n \neq n + n = \phi(I) + \phi(I) \).

(c) \[ \phi : \mathbb{Z}_6 \rightarrow \mathbb{Z}_2 \text{ where for all } x \in \mathbb{Z}_6, \phi(x) = \text{ the remainder of } x \text{ when divided by } 2 \]

Yes. Let \( x, y \in \mathbb{Z}_6 \). Then \( x + y = 6q + r \) for some integers \( q, r \) where \( 0 \leq r < 6 \).
Let \( \psi : \mathbb{Z} \rightarrow \mathbb{Z}_2 \) be the reduction modulo 2 homomorphism. Then
\[
\begin{align*}
\phi(x +_6 y) &= \phi(r) \\
&= \psi(r) \\
&= \psi(x + y - 6q) \\
&= \psi(x + y - 2(3q)) \\
&= \psi(x + y) \\
&= \psi(x) + \psi(y) \\
&= \phi(x) + \phi(y).
\end{align*}
\]
If \( \phi(x) = 0 \), then \( x \) has remainder 0 when divided by 2, so \( \text{ker}(\phi) = \{0, 2, 4\} \). Since \( \text{ker}(\phi) \neq \{0\} \), \( \phi \) is not injective.

(d) \[ \phi : \mathbb{Z}_9 \rightarrow \mathbb{Z}_2 \text{ where for all } x \in \mathbb{Z}_9, \phi(x) = \text{ the remainder of } x \text{ when divided by } 2 \]

No. For example, \( \phi(1 +_9 8) = \phi(0) = 0 \), but \( \phi(1) + \phi(8) = 1 + 2 0 = 1 \).

(e) Let \( F \) be the additive group of all functions mapping \( \mathbb{R} \) into \( \mathbb{R} \) having derivatives of all orders:
\[ \phi : F \rightarrow F \text{ where for all } f \in F, \phi(f) = f'' \]

Yes. Let \( f, g \in F \). Then \( \phi(f + g) = (f + g)'' = (f' + g')' = f'' + g'' = \phi(f) + \phi(g) \). If \( \phi(f) \) is the constant function 0, then \( f \) must be a linear function, so \( \text{ker}(\phi) = \{f \in F \mid f(x) = a + bx \text{ for some } a, b \in \mathbb{R}\} \). Since
3. Let $G$ be a group and $g \in G$. The map $i_g : G \to G$ where $i_g(x) = gxg^{-1}$ for all $x \in G$ is the inner automorphism of $G$ by $g$. Show that $i_g$ is an isomorphism.

Let $g \in G$. If $x, y \in G$, then

$$i_g(xy) = g(xy)g^{-1} = (gxg^{-1})(gyg^{-1}) = i_g(x)i_g(y),$$

so $i_g$ is a homomorphism. Now take the map $i_{g^{-1}} : G \to G$ where $i_{g^{-1}}(x) = g^{-1}xg$ for all $x \in G$. Since

$$i_gi_{g^{-1}}(x) = i_g(g^{-1}xg) = g(g^{-1}xg)g^{-1} = x$$

and

$$i_{g^{-1}}i_g(x) = i_{g^{-1}}(gxg^{-1}) = g^{-1}(gxg^{-1})g = x$$

for all $x \in G$, then $i_{g^{-1}}$ is the inverse of $i_g$, so $i_g$ is bijective. Therefore, $i_g$ is an isomorphism.

4. (§13, #44) Let $\phi : G \to G'$ be a group homomorphism. Show that if $|G|$ is finite, then $|\phi[G]|$ is finite and is a divisor of $|G|$.

Since $\phi$ induces a map from $G$ onto $\phi[G]$, then $|G| \geq |\phi[G]|$. Therefore, since $G$ is finite, so is $\phi[G]$. If $H = \ker(\phi)$, then there is a one-to-one correspondence between the cosets of $H$ in $G$ and $\phi[G]$, so $(G : H) = |\phi[G]|$. Since $G$ is finite, $(G : H) = \frac{|G|}{|H|}$, so $|G| = |\phi[G]| |H|$ and hence $|\phi[G]|$ divides $|G|$.

5. (§13, #45) Let $\phi : G \to G'$ be a group homomorphism. Show that if $|G'|$ is finite, then $|\phi[G]|$ is finite and is a divisor of $|G'|$.

If $|G'|$ is finite, then $\phi[G]$ is also finite since it is a subset of $G'$. By Lagrange, since $\phi[G]$ is a subgroup of $G'$, $|\phi[G]|$ divides $|G'|$.

6. (§13, #47) Show that any group homomorphism $\phi : G \to G'$ where $|G|$ is a prime must either be the trivial homomorphism or a one-to-one map.
9. \((\Ss 8)\) Let \(S\) be a known group to which \(G\) is isomorphic. Describe the possibilities for the kernel of \(\phi\) defined by \(\phi\). If \(\ker(\phi) = 1\), then \(\ker(\phi)\) is the trivial subgroup of \(G\) and hence \(\phi\) is injective. If \(\ker(\phi) = p\), then \(\ker(\phi) = G\), so \(\phi\) sends all elements of \(G\) to \(e'\). In other words, \(\phi\) is the trivial homomorphism. \(\square\)

7. \((\Ss 13, \# 51)\) Let \(G\) be any group and let \(a\) be any element of \(G\). Let \(\phi: \Z \rightarrow G\) be defined by \(\phi(n) = a^n\). Show that \(\phi\) is a homomorphism. Describe the image and the possibilities for the kernel of \(\phi\).

\[\begin{align*}
\text{Proof.} \quad \text{Let } G \text{ be any group and let } a \text{ be any element of } G. \quad \text{Let } \phi: \Z \rightarrow G \text{ be defined by } \phi(n) = a^n. \quad \text{If } n, m \in \Z, \text{ then } \phi(n + m) = a^{n+m} = a^n a^m = \phi(n) \phi(m), \text{ so } \phi \text{ is a homomorphism. The image of } \phi \text{ is } \phi[\Z] = \{\phi(n) \mid n \in \Z\} = \{a^n \mid n \in \Z\}, \text{ in other words, the subgroup of } G \text{ generated by } a. \quad \text{If } a \text{ has order } k, \text{ where } k \text{ is a positive integer, then } \phi(n) = e \text{ implies that } a^n = e, \text{ so } k \text{ must divide } n. \quad \text{Since } n \text{ is a multiple of } k, \ker(\phi) = k\Z. \quad \text{If } a \text{ has infinite order, then } a^n = e \text{ only when } n = 0, \text{ so } \ker(\phi) = \{0\}. \quad \square
\end{align*}\]

8. Let \(G\) be a group. Consider the set \(Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}\). This set is called the center of \(G\). Prove that \(Z(G) \leq G\) and \(Z(G)\) is normal in \(G\).

- Let \(a, b \in Z(G)\). For any \(g \in G\),
  \[
  (ab)g = a(bg) = a(gb) = (ag)b = (ga)b = g(ab),
  \]
  so \(ab \in Z(G)\).
- For any \(g \in G\), \(eg = g = ge\), so \(e \in Z(G)\).
- Let \(a \in Z(G)\). For any \(g \in G\), since \(g^{-1} \in G\), \(ag^{-1} = g^{-1}a\). Taking the inverse of both sides, \(ga^{-1} = a^{-1}g\), so \(a^{-1} \in Z(G)\).

Therefore, \(Z(G)\) is a subgroup of \(G\).

Now let \(a \in G\). By the definition of \(Z(G)\),
\[
 aZ(G) = \{az \mid z \in Z(G)\} = \{za \mid z \in Z(G)\} = Z(G)a,
\]
so \(Z(G)\) is normal in \(G\).

9. \((\Ss 14, \# 24)\) Show that \(A_n\) is a normal subgroup of \(S_n\) and compute \(S_n/A_n\); that is, find a known group to which \(S_n/A_n\) is isomorphic.

\[\begin{align*}
\text{Proof.} \quad \text{If } n = 1, \text{ then } S_1 = \{e\} = A_1. \quad \text{Since } S_1/A_1 \text{ contains the single coset } A_1, \text{ } A_1 \text{ is normal in } S_1 \text{ and } S_1/A_1 \text{ is the trivial group. If } n \geq 2, \text{ then } \|A_n\| = \frac{|S_n|}{2}. \quad \text{Then } (S_n : A_n) = \frac{|S_n|}{|A_n|} = 2. \quad \text{Recall from Homework 6 that any subgroup of index 2 is normal, so } A_n \text{ is a normal subgroup of } S_n. \quad \text{Since } |S_n/A_n| = (S_n : A_n) = 2 \text{ and 2 is prime, } S_n/A_n \text{ is isomorphic to } \Z_2. \quad \square
\end{align*}\]
Alternatively, recall that $\text{sgn}: S_n \to \{\pm1\}$ is a group homomorphism from homework 5. The kernel of $\text{sgn}$ is $A_n$ and the map is surjective, so by the fundamental homomorphism theorem, $S_n/A_n \cong \mathbb{Z}_2$.

10. (§14, #30) Let $H$ be a normal subgroup of $G$, and let $m = (G:H)$. Show that $a^m \in H$ for every $a \in G$.

Proof. Let $H$ be a normal subgroup of $G$, and $m = (G:H)$. Then the collection of left cosets, $G/H$, forms a group of size $m$. For any $a \in G$, $aH \in G/H$ so $(aH)^m = H$. This implies that $a^mH = H$, so $a^m \in H$. \qed