Final Exam – Extra Practice Problems Solutions

The following includes textbook problems and problems from past Math 109 courses to help you prepare for the exam. Since the exam is cumulative, you should still review all past homework problems, lecture notes, discussion notes, previous midterm practice problems, and chapters 12–16, 19, 21, 22, as well as chapters 1–11 of the textbook.

1. Find the coefficient of \(x^3 y^{2015}\) in \((x + y)^{2018}\). You do not need to simplify your answer.

   By the binomial theorem, \((x + y)^{2018} = \sum_{i=0}^{2018} \binom{2018}{i} x^{2018-i} y^i\), so the coefficient of \(x^3 y^{2015}\) is \(\binom{2018}{2015}\).

2. Prove that for any \(n \in \mathbb{Z}^+\), \(\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0\).

   **Proof. One solution:** If \(n = 1\), then \(\sum_{i=0}^{1} (-1)^i \binom{1}{i} = (\binom{1}{0}) - (\binom{1}{1}) = 1 - 1 = 0\). If \(n \geq 2\), then

   \[
   \sum_{i=0}^{n} (-1)^i \binom{n}{i} = (-1)^0 \binom{n}{0} + \sum_{i=1}^{n-1} (-1)^i \binom{n}{i} + (-1)^n \binom{n}{n} \\
   = (-1)^0 \binom{n}{0} + \sum_{i=1}^{n-1} (-1)^i \left[ \binom{n-1}{i} + \binom{n-1}{i-1} \right] + (-1)^n \binom{n}{n} \\
   = (-1)^0 \binom{n-1}{0} + \sum_{i=1}^{n-1} (-1)^i \binom{n-1}{i} + \sum_{i=1}^{n-2} (-1)^{i+1} \binom{n-1}{i} + (-1)^n \binom{n-1}{n-1} \\
   = \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} + \sum_{i=0}^{n-1} (-1)^{i+1} \binom{n-1}{i} \\
   = \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} - \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} = 0.
   \]

   **Another solution:** By the binomial theorem, \((x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i\), so by replacing \(x\) with 1 and \(y\) with \(-1\), we have \(0 = (1 - 1)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^n = \sum_{i=1}^{n} (-1)^i \binom{n}{i}\).

3. Let \(X\) be a denumerable set. Prove that there exists an injection \(f: X \to X\) which is not a surjection.
Proof. Assume $X$ is denumerable. Then there exists a bijection $g: \mathbb{Z}^+ \to X$. Define $f: X \to X$ by $f(x) = g(i+1)$ if $x = g(i)$ (which we can do for each $x \in X$ because $g$ is surjective). Suppose $f(x) = f(y)$ where $x, y \in X$. Then $x = g(i)$ and $y = g(j)$ for some $i, j \in \mathbb{Z}^+$. By the definition of $f$, $g(i+1) = g(j+1)$, which implies that $i+1 = j+1$ since $g$ is injective. Then $i = j$, so $x = g(i) = g(j) = y$ and therefore $f$ is injective. Notice $f$ sends $g(i)$ to $g(i+1)$ for all $i \in \mathbb{Z}^+$, so $g(1) \in X$ has no preimage under $f$ and hence $f$ is not surjective.

4. Recall that the Fibonacci sequence is defined as follows:

$$u_1 = 1, \quad u_2 = 1, \quad u_{n+1} = u_n + u_{n-1} \quad \text{for } n \geq 2.$$ 

Use induction on $n$ to prove that $\gcd(u_{n+1}, u_n) = 1$ for all $n \in \mathbb{Z}^+$.

Proof. We prove by induction on $n$. Base case: If $n = 1$, then $\gcd(u_2, u_1) = \gcd(1, 1) = 1$. Inductive step: Suppose $\gcd(u_{k+1}, u_k) = 1$ for some $k \in \mathbb{Z}^+$. By the division theorem, there exist unique $q, r \in \mathbb{Z}$ such that $u_{k+2} = u_{k+1}q + r$ and $0 \leq r < u_{k+1}$. Since $u_{k+2} = u_{k+1} + u_k$ and $0 < u_k < u_{k+1}$, $r = u_k$. So $\gcd(u_{k+2}, u_{k+1}) = \gcd(u_{k+1}, u_k) = 1$ by the induction hypothesis.

5. Suppose that a positive integer is written in decimal notation as $n = a_k a_{k-1} \ldots a_2 a_1 a_0$ where $0 \leq a_i \leq 9$. Prove that $n$ is divisible by 3 if and only if the sum of its digits $a_k + a_{k-1} + \cdots + a_1 + a_0$ is divisible by 3.

Proof. If $n = a_k a_{k-1} \ldots a_2 a_1 a_0$, then $n = \sum_{i=0}^{k} a_i 10^i$. Then

$$3 \text{ divides } n \iff n \equiv 0 \mod 3$$

$$\iff \sum_{i=0}^{k} a_i 10^i \equiv 0 \mod 3$$

$$\iff \sum_{i=0}^{k} a_i 1^i \equiv 0 \mod 3$$

$$\iff \sum_{i=0}^{k} a_i \equiv 0 \mod 3$$

$$\iff 3 \text{ divides } a_k + a_{k-1} + \cdots + a_1 + a_0.$$

6. Let $m \in \mathbb{Z}^+$. Suppose that $[a_1]_m = [a_2]_m$ and $[b_1]_m = [b_2]_m$. Prove that $[a_1]_m + [b_1]_m = [a_2]_m + [b_2]_m$. This proves that addition in $\mathbb{Z}_m$ is well-defined.
Proof. Suppose that \([a_1]_m = [a_2]_m\) and \([b_1]_m = [b_2]_m\). Since \(a_1 \in [a_2]_m\), \(a_1 \equiv a_2\) mod \(m\), and since \(b_1 \in [b_2]_m\), \(b_1 \equiv b_2\) mod \(m\). This implies that \(a_1 + b_1 \equiv a_2 + b_2\) mod \(m\), so \([a_1 + b_1]_m\) and \([a_2 + b_2]_m\) are equal. Therefore,

\([a_1]_m + [b_1]_m = [a_1 + b_1]_m = [a_2 + b_2]_m = [a_2]_m + [b_2]_m\). \(\square\)

7. Write the multiplication table for \(\mathbb{Z}_5\).

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8. Determine whether the relation on \(X\) is an equivalence relation. Prove your answers.

(a) For \(X = \mathbb{Z}\), define \(a \sim b \iff a + b\) is even.

Proof. Let \(a, b, c \in \mathbb{Z}\).

- \(a + a = 2a\) is even, so \(a \sim a\).
- If \(a \sim b\), then \(a + b\) is even. Since \(b + a = a + b\), \(b \sim a\).
- Suppose that \(a \sim b\) and \(b \sim c\). Then \(a + b\) and \(b + c\) are even, so there exist \(p, q \in \mathbb{Z}\) such that \(a + b = 2p\) and \(b + c = 2q\). This implies that \(a + c = (2p - b) + (2q - b) = 2p + 2q - 2b = 2(p + q - b)\). Since \(a + c\) is even, \(a \sim c\).

Therefore, \(\sim\) is an equivalence relation on \(\mathbb{Z}\). \(\square\)

(b) For \(X = \mathbb{Z}\), define \(a \sim b \iff a + b\) is odd.

Proof. Since \(0 + 0 = 0\) is even, \(0 \not\sim 0\), so \(\sim\) is not reflexive and hence not an equivalence relation on \(\mathbb{Z}\). \(\square\)

(c) For \(X = \mathbb{R}\), define \(a \sim b \iff |a - b| < 1\).

Proof. Notice \(0 \sim \frac{1}{2}\) and \(\frac{1}{2} \sim 1\) because \(|0 - \frac{1}{2}| = \frac{1}{2} < 1\) and \(|\frac{1}{2} - 1| = \frac{1}{2} < 1\). Since \(|0 - 1| = 1 \not\sim 1\), \(0 \not\sim 1\) so \(\sim\) is not transitive and hence not an equivalence relation on \(\mathbb{R}\). \(\square\)

(d) For \(X = \mathbb{R}^2\), define \((a_1, a_2) \sim (b_1, b_2) \iff a_1 = b_1\).
Proof. Let \((a_1, a_2), (b_1, b_2), (c_1, c_2) \in \mathbb{R}^2\).

- \(a_1 = a_1\), so \((a_1, a_2) \sim (a_1, a_2)\).
- If \((a_1, a_2) \sim (b_1, b_2)\), then \(a_1 = b_1\). Since \(b_1 = a_1\), \((b_1, b_2) \sim (a_1, a_2)\).
- Suppose that \((a_1, a_2) \sim (b_1, b_2)\) and \((b_1, b_2) \sim (c_1, c_2)\). Then \(a_1 = b_1\) and \(b_1 = c_1\), so \(a_1 = c_1\) and hence \((a_1, a_2) \sim (c_1, c_2)\).

Therefore, \(\sim\) is an equivalence relation on \(\mathbb{R}^2\). \(\square\)

9. Give an example of a set \(X\) and a relation \(\sim\) on \(X\) such that \(\sim\) reflexive and transitive but not symmetric.

Proof. Let \(X = \mathbb{R}\) and define \(a \sim b \iff a \leq b\). For any \(a \in \mathbb{R}\), \(a \leq a\), so \(a \sim a\). For any \(a, b, c \in \mathbb{R}\), if \(a \sim b\) and \(b \sim c\), then \(a \leq b\) and \(b \leq c\), so \(a \leq c\) and therefore \(a \sim c\). However, \(0 \sim 1\) because \(0 \leq 1\), but \(1 \not\sim 0\) because \(1 \not\leq 0\). So \(\sim\) is a reflexive and transitive relation on \(\mathbb{R}\), but \(\sim\) is not symmetric. \(\square\)

10. Let \(\sim\) be an equivalence relation on a set \(X\). Prove that for any \(a, b \in X\), \(a \sim b\) if and only if \([a]_\sim = [b]_\sim\) (where \([a]_\sim\), \([b]_\sim\) are the equivalence classes of \(a, b\) respectively).

Proof. Let \(\sim\) be an equivalence relation on \(X\) and let \(a, b \in X\).

(\(\Rightarrow\)) Suppose that \(a \sim b\). Let \(x \in [a]_\sim\). Then \(x \sim a\), and by the transitivity of \(\sim\), \(x \sim b\). This means that \(x \in [b]_\sim\).

Similarly, if \(x \in [b]_\sim\), then \(x \sim b\). By the symmetry of \(\sim\), \(b \sim a\), so \(x \sim a\) using the transitive property of \(\sim\). This means that \(x \in [a]_\sim\).

Therefore, \([a]_\sim = [b]_\sim\).

(\(\Leftarrow\)) Suppose that \([a]_\sim = [b]_\sim\). Since \(\sim\) is reflexive, \(a \sim a\) so \(a \in [a]_\sim\). Since \([a]_\sim = [b]_\sim\), \(a \in [b]_\sim\), so \(a \sim b\). \(\square\)