Midterm Exam 1 – Extra Practice Problems

The following includes textbook problems and problems from past Math 109 courses to help you prepare for the exam. However, you should still review all past homework problems, lecture notes, and chapters 1 through 5 of the textbook.

1. Use truth tables to show that each pair of statements are equivalent.
   - \( \neg(P \lor Q) \) and \( (\neg P) \land (\neg Q) \)
   - \( \neg(P \land Q) \) and \( (\neg P) \lor (\neg Q) \)
   - \( \neg(P \Rightarrow Q) \) and \( P \land \neg Q \)
   - \( P \Rightarrow Q \) and \( (\neg P) \lor Q \)
   - \( P \Rightarrow Q \) and \( (\neg Q) \Rightarrow (\neg P) \)
   - \( (P \lor Q) \Rightarrow R \) and \( (P \Rightarrow R) \land (Q \Rightarrow R) \)
   - \( P \Rightarrow (Q \land R) \) and \( (P \Rightarrow Q) \land (P \Rightarrow R) \)
   - \( P \land (Q \lor R) \) and \( (P \land Q) \lor (P \land R) \)

2. For integers \( a, b, \) and \( c \), prove that if \( a \) divides \( b \) and \( a \) divides \( c \), then \( a \) divides \( b + c \).

3. For positive real numbers \( x \) and \( y \), prove that \( \frac{x}{y} + \frac{y}{x} \geq 2 \).

4. For real numbers \( a, b, c, \) and \( d \) such that \( a > b \) and \( c > d \), show that \( ac + bd > ad + bc \).

5. Prove that 0 divides an integer \( a \) if and only if \( a = 0 \).

6. Prove that there do not exist integers \( m \) and \( n \) such that \( 27m + 18n = 100 \).

7. Prove that there is no greatest even integer.

8. Let \( n \) be an integer. Prove that if 3 does not divide \( n^2 \), then 3 does not divide \( n \).

9. Prove that for all positive integers \( m \geq 10 \), \( m^3 \leq 2^m \).

10. Prove that \( \sum_{i=1}^{n} (2i - 1) = n^2 \) for any positive integer \( n \).

11. Prove that \( \sum_{i=1}^{n} i = \frac{n(n + 1)}{2} \) for any positive integer \( n \).

12. Let \( a_1 = 1 \) and \( a_{n+1} = \frac{3a_n + 1}{2a_n + 1} \) for any positive integer \( n \). Prove that
   - For any positive integer \( n \), \( a_n < a_{n+1} \).
   - For any positive integer \( n \), \( a_n < \frac{1 + \sqrt{3}}{2} \).
13. Two people play a game by removing a number of stones from one of two piles set before them. The players take turns and must remove at least one stone during their turn. The player who removes the last stone(s) wins. Prove that if both piles begin with the same number of stones, then the player who goes second has a winning strategy.