Midterm Exam 2 – Extra Practice Problems

The following includes textbook problems and problems from past Math 109 courses to help you prepare for the exam. However, you should still review all past homework problems, lecture notes, and chapters 6 through 11 of the textbook.

1. Prove or disprove the following statements.
   (a) \((\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x + y = 0)\)
   (b) \((\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(x + y = 0)\)
   (c) \((\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(xy = 0)\)
   (d) \((\forall n \in \mathbb{Z}^+)(n \text{ is even or } n \text{ is odd})\)
   (e) \((\forall n \in \mathbb{Z}^+)(n \text{ is even}) \text{ or } (\forall n \in \mathbb{Z}^+)(n \text{ is odd})\)

2. Let \(X\) and \(Y\) be sets. Prove that \(\mathcal{P}(X \cap Y) = \mathcal{P}(X) \cap \mathcal{P}(Y)\).

3. Let \(A, B, C, D\) be sets. Prove that
   (a) \(A \times (B \cup C) = (A \times B) \cup (A \times C)\)
   (b) \((A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)\)

4. Let \(X, Y, Z\) be sets and \(f: X \rightarrow Y, g: Y \rightarrow Z\) be functions.
   (a) Prove that if \(g \circ f: X \rightarrow Z\) is an injection, then \(f\) is an injection. Must \(g\) be injective?
   (b) Prove that if \(g \circ f: X \rightarrow Z\) is a surjection, then \(g\) is a surjection. Must \(f\) be surjective?

5. Let \(f: X \rightarrow Y\) be a function. Prove that \(f\) is injective if and only if \(\overleftarrow{f}\) is injective.

6. Let \(f: X \rightarrow Y\) be a function.
   (a) Prove that for any \(A \in \mathcal{P}(X)\), \(A \subseteq \overleftarrow{f}(\overrightarrow{f}(A))\). If, in addition, \(f\) is injective, prove that \(A = \overleftarrow{f}(\overrightarrow{f}(A))\).
   (b) Prove that \(f\) is injective if and only if \(\overleftarrow{f}\) is surjective.

7. Let \(X, Y\) be subsets of \(\mathbb{N}_n\), where \(n\) is some positive integer. Prove that if \(|X|+|Y| > n\), then \(X \cap Y \neq \emptyset\).

8. Find the number of positive integers less than or equal to 1,000,000 that are neither perfect squares nor perfect cubes.
9. Remove two diagonally opposite corner squares from a chessboard. Prove that the remaining board cannot be covered by tiles consisting of exactly two squares (i.e. $2 \times 1$ tiles).

![Chessboard](image)

10. Suppose that $A$ and $B$ are non-empty finite sets of real numbers such that $A \subseteq B$. Prove that $\min B \leq \min A \leq \max A \leq \max B$. 