1. Prove or disprove the following statements.

(a) \( (\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x + y = 0) \)

**Proof.** True. Let \( x \in \mathbb{R} \). Take \( y = -x \). Then \( x + y = x - x = 0 \).

(b) \( (\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(x + y = 0) \)

**Proof.** False. Let \( y \in \mathbb{R} \). Take \( x = 1 - y \). Then \( x + y = 1 - y + y = 1 \neq 0 \). Therefore, its negation, \( (\forall y \in \mathbb{R})(\exists x \in \mathbb{R})(x + y \neq 0) \), is true.

(c) \( (\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(xy = 0) \)

**Proof.** True. Take \( y = 0 \). Then for all \( x \in \mathbb{R} \), \( xy = x(0) = 0 \).

(d) \( (\forall n \in \mathbb{Z}^+)(n \text{ is even or } n \text{ is odd}) \)

**Proof.** True. Let \( n \in \mathbb{Z}^+ \). Then either \( n \) is even, or \( n \) is not even, which by definition means \( n \) is odd.

(e) \( (\forall n \in \mathbb{Z}^+)(n \text{ is even}) \text{ or } (\forall n \in \mathbb{Z}^+)(n \text{ is odd}) \)

**Proof.** False. Consider the integers 1, which is odd, and 2, which is even. Therefore, its negation, \( (\exists n \in \mathbb{Z}^+)(n \text{ is odd}) \) and \( (\exists n \in \mathbb{Z}^+)(n \text{ is even}) \), is true.

2. Let \( X \) and \( Y \) be sets. Prove that \( \mathcal{P}(X \cap Y) = \mathcal{P}(X) \cap \mathcal{P}(Y) \).

**Proof.** (\( \subseteq \)) Let \( A \in \mathcal{P}(X \cap Y) \). Then \( A \subseteq X \cap Y \). Since \( X \cap Y \subseteq X, Y \), then \( A \subseteq X \) and \( A \subseteq Y \). Therefore, \( A \in \mathcal{P}(X) \) and \( A \in \mathcal{P}(Y) \), so \( A \in \mathcal{P}(X) \cap \mathcal{P}(Y) \).

(\( \supseteq \)) Let \( A \in \mathcal{P}(X) \cap \mathcal{P}(Y) \). Then \( A \in \mathcal{P}(X) \) and \( A \in \mathcal{P}(Y) \). Since \( A \subseteq X \) and \( A \subseteq Y \), then \( A \subseteq X \cap Y \). Therefore, \( A \in \mathcal{P}(X \cap Y) \).

3. Let \( A, B, C, D \) be sets. Prove that

(a) \( A \times (B \cup C) = (A \times B) \cup (A \times C) \)

**Proof.** (\( \subseteq \)) Let \( (x, y) \in A \times (B \cup C) \). Then \( x \in A \) and \( y \in B \cup C \), and so \( y \in B \) or \( y \in C \).

- If \( y \in B \), then \( (x, y) \in A \times B \).
- If \( y \in C \), then \( (x, y) \in A \times C \).

Since \( (x, y) \in A \times B \) or \( (x, y) \in A \times C \), \( (x, y) \in (A \times B) \cup (A \times C) \).

(\( \supseteq \)) Now let \( (x, y) \in (A \times B) \cup (A \times C) \). Then \( (x, y) \in A \times B \) or \( (x, y) \in A \times C \).
4. Let $X$, $Y$, $Z$ be sets and $f: X \to Y$, $g: Y \to Z$ be functions.

(a) Prove that if $g \circ f: X \to Z$ is an injection, then $f$ is an injection. Must $g$ be injective?

\begin{proof}
Assume that $g \circ f$ is injective. Let $x_1, x_2 \in X$ and suppose that $f(x_1) = f(x_2)$. Taking $g$ of both sides, we have $g(f(x_1)) = g(f(x_2))$, i.e. $(g \circ f)(x_1) = (g \circ f)(x_2)$. Since $g \circ f$ is injective, $x_1 = x_2$, so $f$ is also injective. No, $g$ need not be injective. Take $f: \{a\} \to \{b_1, b_2\}$ where $f(a) = b_1$ and $g: \{b_1, b_2\} \to \{a\}$ where $g(b_1) = a$ and $g(b_2) = a$. Notice $g \circ f: \{a\} \to \{a\}$ where $(g \circ f)(a) = a$ is injective but $g$ is not.
\end{proof}

(b) Prove that if $g \circ f: X \to Z$ is a surjection, then $g$ is a surjection. Must $f$ be surjective?

\begin{proof}
Assume that $g \circ f$ is surjective. Let $z \in Z$. Since $g \circ f$ is surjective, there exists $x \in X$ such that $(g \circ f)(x) = z$. Since $g(f(x)) = z$, where $f(x)$ is some element in $Y$, $g$ is also surjective. No, $f$ need not be surjective. Take $f: \{a\} \to \{b_1, b_2\}$ where $f(a) = b_1$ and $g: \{b_1, b_2\} \to \{a\}$ where $g(b_1) = a$ and $g(b_2) = a$. Notice $g \circ f: \{a\} \to \{a\}$ where $(g \circ f)(a) = a$ is surjective but $f$ is not.
\end{proof}

5. Let $f: X \to Y$ be a function. Prove that $f$ is injective if and only if $\overrightarrow{f}$ is injective.

\begin{proof}
$(\Rightarrow)$ Suppose that $f$ is injective. Let $A_1, A_2 \in \mathcal{P}(X)$ and suppose that $\overrightarrow{f}(A_1) = \overrightarrow{f}(A_2)$. Then $\{f(x) \mid x \in A_1\} = \{f(x) \mid x \in A_2\}$. Let $x_1 \in A_1$. Then $f(x_1) \in \{f(x) \mid x \in A_1\}$, so $f(x_1) \in \{f(x) \mid x \in A_2\}$. This means that $f(x_1) = f(x_2)$ for some $x_2 \in A_2$. Since $f$ is injective, $x_1 = x_2$, so $x_1 \in A_2$ and hence $A_1 \subseteq A_2$. A similar argument shows that $A_2 \subseteq A_1$, so $A_1 = A_2$ and therefore $\overrightarrow{f}$ is also injective.

$(\Leftarrow)$ Suppose that $\overrightarrow{f}$ is injective. Let $x_1, x_2 \in X$ and suppose that $f(x_1) = f(x_2)$. Then $\overrightarrow{f}(\{x_1\}) = \{f(x_1)\} = \{f(x_2)\} = \overrightarrow{f}(\{x_2\})$. Since $\overrightarrow{f}$ is injective, $\{x_1\} = \{x_2\}$.
\end{proof}
6. Let $f: X \to Y$ be a function.

(a) Prove that for any $A \in \mathcal{P}(X)$, $A \subseteq \mathcal{f} (\mathcal{f} (A))$. If, in addition, $f$ is injective, prove that $A = \mathcal{f} (\mathcal{f} (A))$.

**Proof.** Let $A \in \mathcal{P}(X)$. If $a \in A$, then $f(a) \in \mathcal{f} (\mathcal{f} (A))$. Let $C = \mathcal{f} (\mathcal{f} (A))$, so $f(a) \in C$. Therefore, $a \in \mathcal{f} (C) = \mathcal{f} (\mathcal{f} (A))$, so we have $A \subseteq \mathcal{f} (\mathcal{f} (A))$.

Now assume $f$ is an injection. Let $x \in \mathcal{f} (\mathcal{f} (A))$. Then $f(x) \in \mathcal{f} (A)$, so $f(x) = f(a)$ for some $a \in A$. Since $f$ is injective, $x = a$, so $x \in A$. Therefore, $\mathcal{f} (\mathcal{f} (A)) \subseteq A$, and hence $A = \mathcal{f} (\mathcal{f} (A))$. \qed

(b) Prove that $f$ is injective if and only if $\mathcal{f}$ is surjective.

**Proof.** ($\Rightarrow$) Suppose that $f$ is injective. Let $A \in \mathcal{P}(X)$. By part (a), $A = \mathcal{f} (\mathcal{f} (A))$. Notice that $\mathcal{f} (A)$ is a subset of $Y$, so $\mathcal{f}$ is surjective.

($\Leftarrow$) Suppose that $\mathcal{f}$ is a surjection. Let $x_1, x_2 \in X$ and assume that $f(x_1) = f(x_2)$. Since $f$ is surjective and $\{x_1\} \in \mathcal{P}(X)$, there exists $B \in \mathcal{P}(Y)$ such that $\mathcal{f} (B) = \{x_1\}$. This means that $f(x_1) \in B$, so $f(x_2) \in B$. Then $x_2 \in \mathcal{f} (B) = \{x_1\}$, so $x_2 = x_1$. Thus, $f$ is injective. \qed

7. Let $X$, $Y$ be subsets of $\mathbb{N}_n$, where $n$ is some positive integer. Prove that if $|X| + |Y| > n$, then $X \cap Y \neq \emptyset$.

**Proof.** Let $X$, $Y$ be subsets of $\mathbb{N}_n$, where $n$ is some positive integer. Assume that $|X| + |Y| > n$. By the inclusion-exclusion principle, $|X \cup Y| = |X| + |Y| - |X \cap Y|$, so by rearranging terms, $|X \cap Y| = |X| + |Y| - |X \cup Y|$. Since $X$ and $Y$ are subsets of $\mathbb{N}_n$, $X \cup Y$ is also a subset of $\mathbb{N}_n$. This implies that $|X \cup Y| \leq n$, so $-|X \cup Y| \geq n$. Then

$$|X \cap Y| = |X| + |Y| - |X \cup Y|$$

$$\geq |X| + |Y| - n$$

$$> n - n = 0$$

Note that the last line follows from the initial assumption. Since the cardinality of $X \cap Y$ is positive, $X \cap Y$ is nonempty. \qed

8. Find the number of positive integers less than or equal to 1,000,000 that are neither perfect squares nor perfect cubes.
\textbf{Proof.} We first count the number of positive integers less than or equal to 1,000,000 that are perfect squares or perfect cubes.

- \textbf{Perfect square:} \( n^2 \leq 1,000,000 \Rightarrow n \leq 1000 \)
- \textbf{Perfect cube:} \( n^3 \leq 1,000,000 \Rightarrow n \leq 100 \)
- \textbf{Perfect square and perfect cube:} \( n^6 \leq 1,000,000 \Rightarrow n \leq 10 \)

Let \( X = \{ n \in \mathbb{Z}^+ \mid n \leq 1,000,000 \text{ and } n \text{ is a perfect square} \} \) and \( Y = \{ n \in \mathbb{Z}^+ \mid n \leq 1,000,000 \text{ and } n \text{ is a perfect cube} \} \). By the inclusion-exclusion principle,

\[ |X \cup Y| = |X| + |Y| - |X \cap Y| = 1000 + 100 - 10 = 1090. \]

Therefore, the number of positive integers less than or equal to 1,000,000 that are neither perfect squares nor perfect cubes is \( 1,000,000 - 1090 = 998,910 \). \( \square \)

9. Remove two diagonally opposite corner squares from a chessboard. Prove that the remaining board cannot be covered by tiles consisting of exactly two squares (i.e. \( 2 \times 1 \) tiles).

\textit{Proof.} Notice that removing two diagonally opposite corner squares will result in removing two squares of the same color. Without loss of generality, remove the two white corner squares. Suppose, for a contradiction, that the board can be covered with these \( 2 \times 1 \) tiles. Let \( X = \{ \text{all tiles in the covering} \} \) and \( Y = \{ \text{all white squares remaining on the board} \} \). Since each tile covers exactly one black square and one white square, define the function \( f : X \rightarrow Y \) where \( f \) maps each tile to the white square that it covers. However, \( |X| = 31 > 30 = |Y| \), so by the pigeonhole principle, \( f \) is not injective. So there exist two tiles that cover the same white square, which is a contradiction. Therefore, no such covering exists. \( \square \)

10. Suppose that \( A \) and \( B \) are non-empty finite sets of real numbers such that \( A \subseteq B \). Prove that \( \min B \leq \min A \leq \max A \leq \max B \).
Proof. Suppose that $A$ and $B$ are non-empty finite sets of real numbers such that $A \subseteq B$. Since $A$ and $B$ are both non-empty finite sets of real numbers, min $A$, max $A$, min $B$, and max $B$ all exist. Since max $A$ is the maximum of element of $A$ and min $A \in A$, min $A \leq$ max $A$. Since min $B$ is the minimum element of $B$, and min $A$ is in $A$ and hence also in $B$, min $B \leq$ min $A$. Lastly, since max $B$ is the maximum element of $B$, and max $A$ is in $A$ and hence also in $B$, max $A \leq$ max $B$. Therefore, min $B \leq$ min $A \leq$ max $A \leq$ max $B$. $\square$