3. Prove the absorption laws.

(i) \(A \cap (A \cup B) = A\)

**Proof.** (\(\subseteq\)) Let \(x \in A \cap (A \cup B)\). By the definition of intersection, \(x \in A\).

(\(\supseteq\)) Let \(x \in A\). Since \(A \subseteq A \cup B\), \(x \in A \cup B\). Therefore, \(x \in A \cap (A \cup B)\). \(\square\)

(ii) \(A \cup (A \cap B) = A\)

**Proof.** (\(\subseteq\)) Let \(x \in A \cup (A \cap B)\). Then \(x \in A\) or \(x \in A \cap B\).

- If \(x \in A\), then we are done.
- If \(x \in A \cap B\), then \(x \in A\) and \(x \in B\).

In either case, \(x \in A\).

(\(\supseteq\)) Let \(x \in A\). By the definition of union, \(x \in A \cup (A \cap B)\). \(\square\)

8. Given sets \(A, B \in P(X)\), their *symmetric difference* is defined by

\[A \triangle B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)\]

Prove that

(i) the symmetric difference is associative (\((A \triangle B) \triangle C = A \triangle (B \triangle C)\) for all \(A, B, C \in P(X)\)),

**Proof.** Let \(A, B, C \in P(X)\).

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Since \(x \in (A \triangle B) \triangle C\) if and only if \(x \in A \triangle (B \triangle C)\), \((A \triangle B) \triangle C = A \triangle (B \triangle C)\). \(\square\)

(ii) there exists a unique set \(N \in P(X)\) such that \(A \triangle N = A\) for all \(A \in P(X)\)

**Proof.** Take \(N = \emptyset\). Then for any \(A \in P(X)\), \(A \triangle N = (A - N) \cup (N - A) = (A - \emptyset) \cup (\emptyset - A) = A \cup \emptyset = A\).

To show uniqueness, assume that \(N' \in P(X)\) also has the property that for any \(A \in P(X)\), \(A \triangle N' = A\). In particular, when \(A = N\), \(N \triangle N' = N\). Similarly, \(A \triangle N = A\) for all \(A \in P(X)\), so if \(A = N'\), then \(N' \triangle N = N'\). Notice that the symmetric difference is commutative, so it follows that

\[N = N \triangle N' = N' \triangle N = N'.\]
(iii) for each $A \in \mathcal{P}(X)$, there exists a unique $A' \in \mathcal{P}(X)$ such that $A \Delta A' = N$.

**Proof.** Let $A \in \mathcal{P}(X)$ and take $A' = A$. Then

$$A \Delta A' = (A - A') \cup (A' - A) = (A - A) \cup (A - A) = \emptyset \cup \emptyset = \emptyset = N.$$  

For uniqueness, suppose that given $A \in \mathcal{P}(X)$, $A'$ and $A''$ in $\mathcal{P}(X)$ have the property that $A \Delta A' = N$ and $A \Delta A'' = N$, respectively. By the associativity and commutativity of the symmetric difference,

$$A \Delta A' = N$$
$$A'' \Delta (A \Delta A') = A'' \Delta N$$
$$(A'' \Delta A) \Delta A' = A''$$
$$(A \Delta A'') \Delta A' = A''$$
$$N \Delta A' = A''$$
$$A' \Delta N = A''$$
$$A' = A''.$$

(iv) for each $A, B \in \mathcal{P}(X)$, there exists a unique set $C$ such that $A \Delta C = B$.

**Proof.** Let $A, B \in \mathcal{P}(X)$. Take $C = A \Delta B$. Then

$$A \Delta C = A \Delta (A \Delta B) = (A \Delta A) \Delta B = N \Delta B = B \Delta N = B.$$  

To prove uniqueness, suppose there is another set $C' \in \mathcal{P}(X)$ such that $A \Delta C' = B$. Then

$$A \Delta C = A \Delta C'$$
$$A \Delta (A \Delta C) = A \Delta (A \Delta C')$$
$$(A \Delta A) \Delta C = (A \Delta A) \Delta C'$$
$$N \Delta C = N \Delta C'$$
$$C \Delta N = C' \Delta N$$
$$C = C'.$