Problems II, Ex. 11 (p. 117)

Refer to the textbook to see the statements. It’s important to keep in mind that quantifiers are read from left to right, so their order matters. For example, \( \forall x \in X, \exists y \in Y, P(x, y) \) is very different from \( \exists y \in Y, \forall x \in X, P(x, y) \).

(i) This is true. Let \( x \in \mathbb{R} \) and pick \( y = 1 - x \). Then \( x + y = x + (1 - x) = 1 > 0 \).

(ii) This is true. Let \( x \in \mathbb{R} \) and pick \( y = x - 1 \). Then \( x - y = x - (x - 1) = 1 > 0 \).

(iii) This is false, and we will prove the negation \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y \leq 0 \). Let \( x \in \mathbb{R} \) and pick \( y = -x \). Then \( x + y = x + (-x) = 0 \leq 0 \).

Note: We stated at the end that \( 0 \leq 0 \) so that it’s clear to the reader that \( x + y \leq 0 \), as we needed to show.

(iv) This is false, and we will prove the negation \( \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, xy \leq 0 \). Pick \( x = 0 \) and let \( y \in \mathbb{R} \). Then \( xy = 0y = 0 \leq 0 \).

Note: We can call \( x = 0 \) a counterexample to the original (false) statement. So instead of saying that “we will prove the negation”, you can say that “we will show a counterexample”.

(v) This is false, and we will prove the negation \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy \leq 0 \). Let \( x \in \mathbb{R} \) and pick \( y = 0 \). Then \( xy = x0 = 0 \leq 0 \).

Note: Personally, I would not call this a counterexample to the original (false) statement because the negation starts with a universal quantifier (since the original statement started with an existential quantifier).

(vi) This is true. Let \( x \in \mathbb{R} \) and pick \( y = 0 \). Then \( xy = x0 = 0 \geq 0 \).

(vii) This is true. Pick \( x = 0 \) and let \( y \in \mathbb{R} \). Then \( xy = 0y = 0 \geq 0 \).

(viii) This is true. Let \( x \in \mathbb{R} \) and pick \( y = -x \). Then \( x + y = x + (-x) = 0 \), so \( x + y > 0 \) or \( x + y = 0 \).

(ix) This is false, and we will prove the negation \( \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, (x + y \leq 0 \text{ or } x + y \neq 0) \). Pick \( x = 0 \) and let \( y \in \mathbb{R} \). Then \( (x + y = 0 \text{ or } x + y \neq 0) \), so \( (x + y \leq 0 \text{ or } x + y \neq 0) \).

Note: In this proof, it didn’t matter what \( x \in \mathbb{R} \) we picked. But be concrete in your proofs and just pick something, like we did here.

(x) This is true, and we will prove separately each of the two statements in the conjunction.

First, let \( x \in \mathbb{R} \) and pick \( y = 1 - x \). Then \( x + y = x + (1 - x) = 1 > 0 \).

Second, let \( x \in \mathbb{R} \) and pick \( y = -x \). Then \( x + y = x + (-x) = 0 \).

Therefore \((\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y > 0) \) and \((\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 0) \).
Problems II, Ex. 14 (p. 117)

Define functions $f$ and $g : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ and $g(x) = x^2 - 1$. Find the functions $f \circ f$, $f \circ g$, $g \circ f$, $g \circ g$. List the elements of the set $\{x \in \mathbb{R} \mid f \circ g(x) = g \circ f(x)\}$.

Each of $f \circ f$, $f \circ g$, $g \circ f$, and $g \circ g$ is a function from $\mathbb{R} \to \mathbb{R}$, and they are defined by

\[
f \circ f(x) = (x^2)^2 = x^4,
\]
\[
f \circ g(x) = (x^2 - 1)^2
\]
\[
g \circ f(x) = (x^2)^2 - 1 = x^4 - 1,
\]
\[
g \circ g(x) = (x^2 - 1)^2 - 1.
\]

Now define $X = \{x \in \mathbb{R} \mid f \circ g(x) = g \circ f(x)\}$ and let $x \in X$. Then

\[
(x^2 - 1)^2 = x^4 - 1
\]
\[
x^4 - 2x^2 + 1 = x^4 - 1
\]
\[
2x^2 = 2
\]
\[
x^2 = 1.
\]

The only numbers that satisfy this are $-1$ and $1$. Therefore $X = \{-1, 1\}$. 

Problems II, Ex. 15 (p. 117)

Given $A \in \mathcal{P}(X)$ define the characteristic function $\chi_A : X \to \{0, 1\}$ by

$$\chi_A(x) = \begin{cases} 
0 & \text{if } x \notin A \\
1 & \text{if } x \in A.
\end{cases}$$

Suppose that $A$ and $B$ are subsets of $X$.

(i) Prove that the function $x \mapsto \chi_A(x)\chi_B(x)$ (multiplication of integers) is the characteristic function of the intersection $A \cap B$.

(ii) Find the subset $C$ whose characteristic function is given by $\chi_C(x) = \chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x)$.

(i) **Proof.** Define the function $f : X \to \{0, 1\}$ by $f(x) = \chi_A(x)\chi_B(x)$. We will show that $f = \chi_{A \cap B}$.

Let $x \in X$. If $x \in A \cap B$, then $x \in A$ and $x \in B$, so $f(x) = 1 \cdot 1 = 1 = \chi_{A \cap B}(x)$. Otherwise, if $x \notin A \cap B$, then $x \notin A$ or $x \notin B$, so $\chi_A(x) = 0$ or $\chi_B(x) = 0$, thus $f(x) = 0 = \chi_{A \cap B}(x)$.

In all cases, $f(x) = \chi_{A \cap B}(x)$, so $f = \chi_{A \cap B}$. ■

(ii) This looks similar to the inclusion–exclusion principle, so we can guess that $C = A \cup B$. Let’s prove that this is correct.

**Proof.** Pick $C = A \cup B$ and let $x \in X$. Then there are four cases: (1) $x \in A$ and $x \in B$, (2) $x \in A$ and $x \notin B$, (3) $x \notin A$ and $x \in B$, and (4) $x \notin A$ and $x \notin B$. Notice that

$$\chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x) = \begin{cases} 
1 + 1 - 1 \cdot 1 = 1 & \text{if } x \in A \text{ and } x \in B, \\
1 + 0 - 1 \cdot 0 = 1 & \text{if } x \in A \text{ and } x \notin B, \\
0 + 1 - 0 \cdot 1 = 1 & \text{if } x \notin A \text{ and } x \in B, \\
0 + 0 - 0 \cdot 0 = 0 & \text{if } x \notin A \text{ and } x \notin B
\end{cases}
$$

$$= \begin{cases} 
1 & \text{if } x \in A \cup B, \\
0 & \text{if } x \notin A \cup B
\end{cases}
= \chi_C(x),$$

and we are done. ■