For some problems, several sample proofs are given. Problems 1 and 9–13 are appended at the end as hand-written pages.

**Problem 2.**

For integers $a$, $b$, and $c$, prove that if $a$ divides $b$ and $a$ divides $c$, then $a$ divides $b + c$.

**Proof.** Let $a$, $b$, and $c$ be integers and suppose that $a$ divides $b$ and $a$ divides $c$. Then by the definition of divides, there are integers $p$ and $q$ such that $b = ap$ and $c = aq$. So

$$b + c = ap + aq = a(p + q),$$

and notice that $p + q$ is an integer. Therefore $a$ divides $b + c$ by definition. ■
Problem 3.

For positive real numbers $x$ and $y$, prove that $x/y + y/x \geq 2$.

Proof (1). Let $x$ and $y$ be positive real numbers. We know that $a^2 \geq 0$ for any real number $a$, so

$$(x - y)^2 \geq 0.$$  

Now, distributivity gives

$$x^2 - 2xy + y^2 \geq 0,$$

then by the addition law for inequalities (for all real numbers $a, b, c$, we have $a > b$ if and only if $a + c > b + c$),

$$x^2 + y^2 \geq 2xy.$$  

Finally, recall the multiplication law for inequalities, which states that for all real numbers $a, b, c$ with $a > 0$, we have $b > c$ if and only if $ab > ac$. Since $x > 0$ and $y > 0$, they have positive multiplicative inverses, so use the multiplication law twice to arrive at our desired inequality

$$\frac{x}{y} + \frac{y}{x} \geq 2.$$  

Note: You don’t have to name every property (e.g., addition/multiplication law, distributivity, etc.) you use in your proofs unless the problem asks for it. But each step must be present and clear. In particular, if you divide by a number, make sure you state that the number is nonzero, and if you multiply an inequality by a number, make sure you state whether the number is positive or negative (or nonnegative or nonpositive) because the direction of the inequality depends on this.

Proof (2). Let $x$ and $y$ be positive real numbers. We will work backward. Notice the following sequence of backward implications:

$$\frac{x}{y} + \frac{y}{x} \geq 2 \iff x^2 + y^2 \geq 2xy \text{ by the multiplication law, since } x > 0 \text{ and } y > 0.$$

$$\iff x^2 + y^2 - 2xy \geq 0 \quad \text{by the addition law.}$$

$$\iff (x - y)^2 \geq 0 \quad \text{by distributivity.}$$

We know that $(x - y)^2 \geq 0$ because the square of any real number is nonnegative, so we are done.

Note: A quick way to prove the inequality $a^2 \geq 0$ for any real number $a$ is to use the multiplication law with three cases: Case 1) if $a > 0$, then $a^2 > 0$ by the multiplication law, Case 2) if $a < 0$, then $a^2 > 0$ again by the multiplication law, and Case 3) if $a = 0$, then $a^2 = 0$. In all cases $a^2 \geq 0$.  

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Problem 4.

For real numbers $a, b, c,$ and $d$ such that $a > b$ and $c > d$, show that $ac + bd > ad + bc$.

**Proof (1).** Let $a, b, c,$ and $d$ be real numbers and suppose that $a > b$ and $c > d$. Then $c - d > 0$, so multiply each side of the inequality $a > b$ by $(c - d)$ to get

\[
\begin{align*}
    a(c - d) &> b(c - d) \\
    ac - ad &> bc - bd \\
    ac + bd &> ad + bc.
\end{align*}
\]

**Proof (2).** Let $a, b, c,$ and $d$ be real numbers and suppose that $a > b$ and $c > d$. We will work backward. Notice the following sequence of backward implications:

\[
\begin{align*}
    ac + bd &> ad + bc \\
    &\iff ac - ad > bc - bd \\
    &\iff a(c - d) > b(c - d) \\
    &\iff a > b \quad \text{because } c - d > 0
\end{align*}
\]

We were given that $a > b$, so we are done.

Problem 5.

Prove that 0 divides an integer $a$ if and only if $a = 0$.

**Proof.** We will prove separately the forward and backward implications.

First, if 0 divides an integer $a$, then $0q = a$ for some integer $q$, so $0 = a$.

Second, if $a = 0$, then $0 \times 0 = a$, so 0 divides $a$.

Therefore 0 divides an integer $a$ if and only if $a = 0$.

Problem 6.

Prove that there do not exist integers $m$ and $n$ such that $27m + 18n = 100$.

**Proof (1).** Notice that $27m + 18n = 9(3m + 2n)$, but 9 does not divide 100, so there cannot be such $m$ and $n$.

**Proof (2).** Assume for contradiction that there are integers $m$ and $n$ such that $27m + 18n = 100$. Then $27m + 18n - 99 = 1$, so $9(3m + 2n - 11) = 1$ where $3m + 2n - 11$ must be positive. But now $9(3m + 2n - 11) \geq 9(1) > 1$, which is a contradiction. Therefore our assumption was false; instead, there cannot be such integers $m$ and $n$.
Problem 7.

Prove that there is no greatest even integer.

Proof. Assume for contradiction that there is a greatest even integer. Call it $n$. Then $n = 2q$ for some integer $q$, so $n + 2 = 2(q + 1)$, meaning that $n + 2$ is also an even integer. But since $n + 2 > n$, we now have $n + 2$ being an even integer greater than $n$, which contradicts $n$ being the greatest even integer. Therefore our assumption was false; instead, there is no greatest even integer.

Problem 8.

Let $n$ be an integer. Prove that if 3 does not divide $n^2$, then 3 does not divide $n$.

Proof (1). We will prove the contrapositive.

Let $n$ be an integer and suppose that 3 divides $n$. Then $n = 3q$ for some integer $q$, so $n^2 = (3q)^2 = 9q^2 = 3(3q^2)$, which means that 3 divides $n^2$.

Proof (2). Let $n$ be an integer and suppose that 3 does not divide $n^2$. Assume for contradiction that 3 divides $n$. Then $n = 3q$ for some integer $q$, so $n^2 = (3q)^2 = 9q^2 = 3(3q^2)$, which means that 3 divides $n^2$. But this contradicts the fact that 3 does not divide $n^2$. Therefore our assumption was false; instead, 3 does not divide $n$. 

■
Problem 9

Proof by induction on $n$.

Base case: For $n = 10$,

\[ n^3 = 1000, \quad 2^n = 1024 \]

So $n^3 \leq 2^n$. \(\checkmark\)

Inductive step:

Suppose that $k^3 \leq 2^k$ for some integer $k \geq 10$.

Then $(k+1)^3 = k^3 + 3k^2 + 3k + 1$

\[ < k^3 + 7k^2 \text{ because } k^2 > k > 1 \]

\[ < k^3 + k^3 \text{ because } k > 7 \]

\[ \leq 2^k + 2^k \]

\[ = 2^{k+1} \]

Thus $(k+1)^3 \leq 2^{k+1}$, and this completes our proof. 

\(\square\)
Problem 10.

Proof by induction on \( n \).

Base case: For \( n = 1 \),

\[
\text{LHS} = \sum_{i=1}^{1} (2i-1) \quad \text{RHS} = 1^2
\]

\[
= 2(1) - 1
= 1
\]

Thus \( \text{LHS} = \text{RHS} \). \( \checkmark \)

Inductive step:

Suppose that \( \sum_{i=1}^{k} (2i-1) = k^2 \) for some positive integer \( k \).

Then

\[
\sum_{i=1}^{k+1} (2i-1) = \sum_{i=1}^{k} (2i-1) + (2(k+1)-1)
= k^2 + 2k + 1
= (k+1)^2
\]
Problem 11.

Proof by induction on $n$.

Base case: For $n = 1$,

$$\text{LHS} = \sum_{i=1}^{1} i = 1$$

$$\text{RHS} = \frac{1(1+1)}{2} = 1$$

Thus $\text{LHS} = \text{RHS}$. \(\checkmark\)

Inductive step:

Suppose that $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ for some positive integer $k$.

Then $\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)$

$$= \frac{k(k+1)}{2} + k + 1$$

$$= \frac{(k+1)(k+2)}{2}$$

//
Problem 12.

Proof by induction on \( n \).

Let \( a_1 = 1 \) and \( a_{n+1} = \frac{3a_n + 1}{2a_n + 1} \) for any positive integer \( n \).

Base case: For \( n=1 \),

\[
a_1 = 1 \quad \text{and} \quad a_2 = \frac{3(1) + 1}{2(1) + 1} = \frac{4}{3},
\]

so \( a_1 < a_2 \), and \( a_1 < \frac{1 + \sqrt{3}}{2} \). \( \checkmark \)

Inductive step:

Suppose that \( a_k < a_{k+1} \) and \( a_k < \frac{1 + \sqrt{3}}{2} \) for some positive integer \( k \).

Then

\[
\frac{a_{k+2}}{a_{k+1}} = \frac{(3a_{k+1} + 1)(2a_k + 1)}{(2a_{k+1} + 1)(3a_k + 1)}
\]

\[
= \frac{6a_{k+1}a_k + 3a_{k+1} + 2a_k + 1}{6a_{k+1}a_k + 2a_{k+1} + 3a_k + 1}
\]

> 1 because \( 3a_{k+1} + 2a_k = 2a_{k+1} + 3a_k + (a_{k+1} - a_k) \)

> 2a_{k+1} + 3a_k

so \( a_{k+2} > a_{k+1} \).

[Continued on next page.]
Now we need to show that $a_{k+1} < \frac{1+\sqrt{3}}{2}$.

We will work backward. Notice the following sequence of backward implications:

\[
 a_{k+1} < \frac{1+\sqrt{3}}{2} \iff \frac{3a_k + 1}{2a_k + 1} < \frac{1+\sqrt{3}}{2}
\]

\[
 \iff 2(3a_k + 1) < (1+\sqrt{3})(2a_k + 1)
\]

\[
 \iff 6a_k + 2 < 2(1+\sqrt{3})a_k + 1 + \sqrt{3}
\]

\[
 \iff 6a_k - 2(1+\sqrt{3})a_k < 1 + \sqrt{3} - 2
\]

\[
 \iff a_k (4 - 2\sqrt{3}) < \sqrt{3} - 1
\]

\[
 \iff a_k < \frac{\sqrt{3} - 1}{4 - 2\sqrt{3}}
\]

\[
 = \frac{(\sqrt{3} - 1)(4 + 2\sqrt{3})}{16 - 12}
\]

\[
 = \frac{4\sqrt{3} - 4 + 6 - 2\sqrt{3}}{4}
\]

\[
 = \frac{\sqrt{3} + 1}{2}
\]

We know that $a_k < \frac{\sqrt{3} + 1}{2}$ because that was part of the inductive hypothesis. Thus $a_{k+1} < \frac{1+\sqrt{3}}{2}$, which finishes the inductive step, and we are done.
Problem 13.

Let \( n \) be the number of stones in each of the two piles at the start of the game.

Proof by strong induction on \( n \).

Base case: For \( n = 1 \), the first player must choose one of the two piles and remove the stone there. Thus the second player will win on their first turn by removing the stone from the other pile.

Inductive step:

Suppose that for some positive integer \( k \), the second player has a winning strategy for every game that begins with \( n \) stones in each of the two piles, where \( n \) is any positive integer such that \( n \leq k \).

Now consider a game that begins with \( k+1 \) stones in each of the two piles.

The first player must begin by removing stones from one of the two piles, so suppose that they remove \( q \) stones, where \( q \) is a positive integer such that \( q \leq k+1 \).

Then the second player can remove \( q \) stones from the other pile. If \( q = k+1 \), then the second player wins immediately. Otherwise, if \( q < k+1 \), then it is now the first player's turn in a game where there are \( k+1-q \leq k \) stones in each pile and the second player already has a winning strategy for that. Thus the second player has a winning strategy for a game that begins with \( k+1 \) stones in each pile.
Problem 1.

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| (1) | (1) | (2) | (2) |

Columns marked (1) are identical, so \(\neg(p v q)\) and \((\neg p) \land (\neg q)\) are equivalent. Similarly for (2), \(\neg(p \land q)\) and \((\neg p) \lor (\neg q)\) are equivalent.

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Same notation as above.

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| (6) | (6) |

Last two equivalences left as exercises for the reader. (i.e., it’s a pain to draw these)