1. Prove that $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for any positive integer $n$.

2. Determine whether each statement is true or false. Justify your answers.
   
   (a) $(\forall x, y \in \mathbb{R})(\exists z \in \mathbb{R})(x + z = y)$
   
   (b) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(xy = 1)$
   
   (c) $(\forall sets X, Y)(X \times Y = Y \times X)$
   
   (d) $(\forall sets X, Y, Z)(X - (Y \cup Z) = (X - Y) \cap (X - Z))$
   
   (e) Assume $X$ and $Y$ are finite sets. Then $(\forall f : X \to Y)(|X| > |Y| \Rightarrow (\exists x_1, x_2 \in X)(x_1 \neq x_2 \land f(x_1) = f(x_2)))$
   
   (f) $(\forall m, n \in \mathbb{Z})(\exists p \in \mathbb{Z})(m = 2p+1) \land (\exists q \in \mathbb{Z})(n = 2q+1) \Rightarrow (\exists r \in \mathbb{Z})(m+n = 2r)$

3. Let $X$ be a denumerable set. Prove that there exists an injection $f : X \to X$ which is not a surjection.

4. Suppose that a positive integer is written in decimal notation as $n = a_k a_{k-1} \ldots a_2 a_1 a_0$ where $0 \leq a_i \leq 9$. Prove that $n$ is divisible by 3 if and only if the sum of its digits $a_k + a_{k-1} + \cdots + a_1 + a_0$ is divisible by 3.

5. Determine whether the relation on $X$ is an equivalence relation. Prove your answers. For those which are equivalence relations, describe the equivalence classes.
   
   (i) For $X = \mathbb{Z}$, define $a \sim b \iff ab \neq 0$.
   
   (ii) For $X = \mathbb{Z}$, define $a \sim b \iff ab \geq 0$.

   (iii) For $X = \mathbb{Z}^+$, define $a \sim b \iff ab > 0$.

   (iv) For $X = \mathbb{Z} - \{0\}$, define $a \sim b \iff ab > 0$.

   (v) For $X = \mathbb{Z}^+$, define $a \sim b \iff ab < 0$.

   (vi) For $X = \mathbb{Z} - \{0\}$, define $a \sim b \iff ab < 0$.

   (a) For $X = \mathbb{Z}$, define $a \sim b \iff a + b$ is even.

   (b) For $X = \mathbb{Z}$, define $a \sim b \iff a + b$ is odd.

   (c) For $X = \mathbb{R}^2$, define $(a_1, a_2) \sim (b_1, b_2) \iff a_1^2 + a_2^2 = b_1^2 + b_2^2$.

   (d) For $X = \mathbb{R}$, define $a \sim b \iff a - b \in \mathbb{Z}$.

6. Give an example of a set $X$ and a relation $\sim$ on $X$ such that $\sim$ is reflexive and transitive but not symmetric.

7. Let $\sim$ be an equivalence relation on a set $X$. Prove that for any $a, b \in X$, $a \not\sim b$ if and only if $[a]_\sim \cap [b]_\sim = \emptyset$ (where $[a]_\sim, [b]_\sim$ are the equivalence classes of $a, b$, respectively).
8. Let $m \in \mathbb{Z}^+$. Let $[a]_m$ denote the equivalence class of $a \in \mathbb{Z}$ for the equivalence relation on $\mathbb{Z}$ given by congruence modulo $m$. Define addition on the set of equivalence classes by $[a]_m + [b]_m = [a + b]_m$. Prove that this operation is well-defined, that is, if $[a_1]_m = [a_2]_m$ and $[b_1]_m = [b_2]_m$, then $[a_1]_m + [b_1]_m = [a_2]_m + [b_2]_m$ for any $a_1, a_2, b_1, b_2 \in \mathbb{Z}$. 