Final Exam – Practice Problems Solutions

1. Prove that $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for any positive integer $n$.

Proof. We prove by induction on $n$.

Base case: If $n = 1$, then $\sum_{i=1}^{1} i^3 = 1^3 = 1 = \frac{1^2(1+1)^2}{4}$.

Inductive step: Suppose that $\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$ where $k$ is a positive integer. Then

$$
\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3
= \frac{k^2(k+1)^2}{4} + (k+1)^3
= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4}
= \frac{(k+1)^2[k^2 + 4(k+1)]}{4}
= \frac{(k+1)^2(k^2 + 4(k+1))}{4}
= \frac{(k+1)^2(k+2)^2}{4}.
$$

Therefore, $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for any positive integer $n$. \[ \square \]

2. Determine whether each statement is true or false. Justify your answers.

(a) $(\forall x, y \in \mathbb{R})(\exists z \in \mathbb{R})(x + z = y)$

Proof. True. Let $x, y \in \mathbb{R}$. Take the real number $z = y - x$. Then $x + z = x + (y - x) = y$. \[ \square \]

(b) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(xy = 1)$

Proof. False. Take $x = 0$. Then for all $y \in \mathbb{R}$, $xy = 0y = 0 \neq 1$. \[ \square \]

(c) $(\forall \text{ sets } X,Y)(X \times Y = Y \times X)$
4. Suppose that a positive integer is written in decimal notation as $n = a_k a_{k-1} \ldots a_2 a_1 a_0$ where $0 \leq a_i \leq 9$. Prove that $n$ is divisible by 3 if and only if the sum of its digits $a_k + a_{k-1} + \cdots + a_1 + a_0$ is divisible by 3.
Proof. If \( n = a_k a_{k-1} \ldots a_1 a_0 \), then \( n = \sum_{i=0}^{k} a_i 10^i \). Then

\[
3 \text{ divides } n \iff n \equiv 0 \mod 3 \\
\iff \sum_{i=0}^{k} a_i 10^i \equiv 0 \mod 3 \\
\iff \sum_{i=0}^{k} a_i 1 \equiv 0 \mod 3 \\
\iff 3 \text{ divides } a_k + a_{k-1} + \cdots + a_1 + a_0.
\]

5. Determine whether the relation on \( X \) is an equivalence relation. Prove your answers. For those which are equivalence relations, describe the equivalence classes.

(i) For \( X = \mathbb{Z} \), define \( a \sim b \iff ab \neq 0 \).

- \( \sim \) is not reflexive. When \( a = 0 \), \( aa = 0(0) = 0 \).
- \( \sim \) is symmetric. For any \( a, b \in \mathbb{Z} \), if \( ab \neq 0 \), then \( ba = ab \neq 0 \).
- \( \sim \) is transitive. Let \( a, b, c \in \mathbb{Z} \) and suppose that \( ab \neq 0 \) and \( bc \neq 0 \). For a contradiction, suppose that \( ac = 0 \). Then \( a = 0 \) or \( c = 0 \), but this would imply that \( ab = 0 \) or \( bc = 0 \). Therefore, \( ac \neq 0 \).

Since \( \sim \) is not reflexive, \( \sim \) is not an equivalence relation.

(ii) For \( X = \mathbb{Z} \), define \( a \sim b \iff ab \geq 0 \).

- \( \sim \) is reflexive. For any \( a \in \mathbb{Z} \), \( aa = a^2 \geq 0 \).
- \( \sim \) is symmetric. For any \( a, b \in \mathbb{Z} \), if \( ab \geq 0 \), then \( ba = ab \geq 0 \).
- \( \sim \) is not transitive. Take \( a = -1, b = 0, c = 1 \). Then \( ab = 0 \geq 0 \) and \( bc = 0 \geq 0 \), but \( ac = -1 \not\geq 0 \).

Since \( \sim \) is not transitive, \( \sim \) is not an equivalence relation.

(iii) For \( X = \mathbb{Z}^+ \), define \( a \sim b \iff ab > 0 \).

- \( \sim \) is reflexive. For any \( a \in \mathbb{Z}^+ \), \( aa = a^2 > 0 \) since the product of two positive integers is a positive integer.
- \( \sim \) is symmetric. For any \( a, b \in \mathbb{Z}^+ \), if \( ab > 0 \), then \( ba > 0 \) since the product of two positive integers is a positive integer.
- \( \sim \) is transitive. Let \( a, b, c \in \mathbb{Z}^+ \). Suppose that \( ab > 0 \) and \( bc > 0 \). Then \( ac > 0 \) since the product of two positive integers is a positive integer.

Therefore, \( \sim \) is an equivalence relation. There is exactly one equivalence class:
(iv) For $X = \mathbb{Z} - \{0\}$, define $a \sim b \iff ab > 0$.

- $\sim$ is reflexive. For any $a \in \mathbb{Z} - \{0\}$, $aa = a^2 > 0$ since $a \neq 0$.
- $\sim$ is symmetric. For any $a, b \in \mathbb{Z} - \{0\}$, if $ab > 0$, then $ba = ab > 0$.
- $\sim$ is transitive. Let $a, b, c \in \mathbb{Z} - \{0\}$. Suppose that $ab > 0$ and $bc > 0$. If $a > 0$, then $b$ must be positive, which implies that $c$ is also positive. Therefore, $ac > 0$. If $a < 0$, then $b$ must be negative, which implies that $c$ is also negative. Therefore, $ac > 0$.

Therefore, $\sim$ is an equivalence relation. There are two equivalence classes: $\mathbb{Z}^+$ and $\{a \in \mathbb{Z} \mid a < 0\}$.

(v) For $X = \mathbb{Z}^+$, define $a \sim b \iff ab < 0$.

- $\sim$ is not reflexive. When $a = 1$, $aa = 1(1) = 1 \neq 0$.
- $\sim$ is symmetric. For any $a, b \in \mathbb{Z}^+$, if $ab < 0$, then $ba < 0$. (This statement is true because the premise is always false.)
- $\sim$ is transitive. For any $a, b, c \in \mathbb{Z}^+$, if $ab < 0$ and $bc < 0$, then $ac < 0$. (This statement is true because the premise is always false.)

Since $\sim$ is not reflexive, $\sim$ is not an equivalence relation.

(vi) For $X = \mathbb{Z} - \{0\}$, define $a \sim b \iff ab < 0$.

- $\sim$ is not reflexive. When $a = 1$, $aa = 1(1) = 1 \neq 0$.
- $\sim$ is symmetric. For any $a, b \in \mathbb{Z} - \{0\}$, if $ab < 0$, then $ba < 0$.
- $\sim$ is not transitive. Take $a = -1, b = 1, c = -1$. Then $ab = -1 < 0$ and $bc = -1 < 0$, but $ac = 1 \neq 0$.

Since $\sim$ is neither reflexive nor transitive, $\sim$ is not an equivalence relation.

(a) For $X = \mathbb{Z}$, define $a \sim b \iff a + b$ is even.

- $\sim$ is reflexive: For any $a \in \mathbb{Z}$, $a + a = 2a$ is even, so $a \sim a$.
- $\sim$ is symmetric: For any $a, b \in \mathbb{Z}$, if $a \sim b$, then $a + b$ is even. Since $b + a = a + b$, $b \sim a$.
- $\sim$ is transitive: Let $a, b, c \in \mathbb{Z}$. Suppose that $a \sim b$ and $b \sim c$. Then $a + b$ and $b + c$ are even, so there exist $p, q \in \mathbb{Z}$ such that $a + b = 2p$ and $b + c = 2q$. This implies that $a + c = (2p - b) + (2q - b) = 2p + 2q - 2b = 2(p + q - b)$. Since $a + c$ is even, $a \sim c$.

Therefore, $\sim$ is an equivalence relation on $\mathbb{Z}$. The equivalence classes are $\{2k \mid k \in \mathbb{Z}\}$ and $\{2k + 1 \mid k \in \mathbb{Z}\}$.

(b) For $X = \mathbb{Z}$, define $a \sim b \iff a + b$ is odd.
6. Give an example of a set $X$ and a relation $\sim$ on $X$ such that $\sim$ is reflexive and transitive but not symmetric.

**Proof.** Let $X = \mathbb{R}$ and define $a \sim b \iff a \leq b$. For any $a \in \mathbb{R}$, $a \leq a$, so $a \sim a$. For any $a, b, c \in \mathbb{R}$, if $a \sim b$ and $b \sim c$, then $a \leq b$ and $b \leq c$, so $a \leq c$ and therefore $a \sim c$. However, $0 \sim 1$ because $0 \leq 1$, but $1 \not\sim 0$ because $1 \not\leq 0$. So $\sim$ is a reflexive and transitive relation on $\mathbb{R}$, but $\sim$ is not symmetric. \qed

7. Let $\sim$ be an equivalence relation on a set $X$. Prove that for any $a, b \in X$, $a \not\sim b$ if and only if $[a]_\sim \cap [b]_\sim = \emptyset$ (where $[a]_\sim$, $[b]_\sim$ are the equivalence classes of $a$, $b$, respectively).
Proof. Let \( \sim \) be an equivalence relation on \( X \) and let \( a, b \in X \).

(\( \Rightarrow \)) Suppose that \( a \not\sim b \). For a contradiction, suppose that \( [a]_\sim \cap [b]_\sim \neq \emptyset \). Then there exists \( x \in X \) such that \( x \in [a]_\sim \) and \( x \in [b]_\sim \), so \( x \sim a \) and \( x \sim b \). By the symmetric property, \( a \sim x \), so by transitivity, \( a \sim b \), which is a contradiction. Therefore, \( [a]_\sim \cap [b]_\sim = \emptyset \).

(\( \Leftarrow \)) Suppose that \( [a]_\sim \cap [b]_\sim = \emptyset \). For a contradiction, suppose that \( a \sim b \). By the reflexive property, \( a \sim a \), so \( a \in [a]_\sim \). Since \( a \sim b \), \( a \in [b]_\sim \), so \( a \in [a]_\sim \cap [b]_\sim \), which is a contradiction. Therefore, \( a \not\sim b \).

8. Let \( m \in \mathbb{Z}^+ \). Let \([a]_m\) denote the equivalence class of \( a \in \mathbb{Z} \) for the equivalence relation on \( \mathbb{Z} \) given by congruence modulo \( m \). Define addition on the set of equivalence classes by \([a]_m + [b]_m = [a + b]_m\). Prove that this operation is well-defined, that is, if \([a]_m = [a']_m \) and \([b]_m = [b']_m\), then \([a]_m + [b]_m = [a']_m + [b']_m\) for any \( a, a', b, b' \in \mathbb{Z} \).

Proof. Suppose that \([a]_m = [a']_m\) and \([b]_m = [b']_m\). Since \( a_1 \in [a]_m \), \( a_1 \equiv a_2 \mod m \), and since \( b_1 \in [b]_m \), \( b_1 \equiv b_2 \mod m \). This implies that \( a_1 + b_1 \equiv a_2 + b_2 \mod m \), so \( a_1 + b_1 \in [a_2 + b_2]_m \). Congruence classes modulo \( m \) are either equal or disjoint, so \([a_1 + b_1]_m \) and \([a_2 + b_2]_m \) are equal. Therefore,

\[ [a_1]_m + [b_1]_m = [a_1 + b_1]_m = [a_2 + b_2]_m = [a_2]_m + [b_2]_m. \]