Homework 5 Solutions and Rubric

Point distribution (out of 20 points in total):

- Problem 11.2 – 4 points.
- Problem 12.2 – 3 points.
- Problem III.14 – 4 points.
- \[ \sum_{i=0}^{n} \binom{n}{i} = \sum_{i=0}^{n} \binom{n}{i} \] – 4 points.
- Completeness – 5 points. This is for attempting the remaining 5 problems.

See the back of the book for the remaining solutions.

Problems III

14. For \( n \in \mathbb{Z}^+ \), suppose that \( A \subseteq \mathbb{N}_{2n} \) and \( |A| = n + 1 \). Prove that \( A \) contains a pair of distinct integers \( a, b \) such that \( a \) divides \( b \).

[Let \( f(a) \) be the greatest odd integer which divides \( a \) and apply the pigeonhole principle to \( f \).]

**Proof.** Let \( n \in \mathbb{Z}^+ \). Suppose that \( A \subseteq \mathbb{N}_{2n} \) and \( |A| = n + 1 \). For any \( a \in A \), a positive odd integer divisor of \( a \) must live in the set \( B = \{ x \in \mathbb{N}_{2n} \mid x \text{ is odd} \} = \{ 2k + 1 \mid k \in \mathbb{Z}^+, 0 \leq k \leq n - 1 \} \). Define the function \( f : A \rightarrow B \) where, for any \( a \in A \), \( f(a) \) is the greatest odd integer which divides \( a \). Since \( |A| = n + 1 > n = |B| \), by the pigeonhole principle, \( f \) is not injective. This means that there exist distinct \( a, b \in A \) such that \( f(a) = f(b) \).

Since \( f(a) \) and \( f(b) \) are the greatest odd integers which divide \( a \) and \( b \), respectively, then we can write \( a = 2^s f(a) \) and \( b = 2^t f(b) \) for some nonnegative integers \( s, t \).

Without loss of generality, suppose \( t \geq s \). Then

\[
b = 2^t f(b) = 2^{t-s} 2^s f(b) = 2^{t-s} 2^s f(a) = 2^{t-s} a.
\]

Since \( t \geq s \), \( t - s \) is some nonnegative integer, so \( 2^{t-s} \in \mathbb{Z} \). Therefore, \( a \) divides \( b \). \( \square \)

19. Prove Leibniz’s rule for higher order derivatives of products,

\[
\frac{d^n(uv)}{dx^n} = \sum_{i=0}^{n} \binom{n}{i} \frac{d^i u}{dx^i} \frac{d^{n-i} v}{dx^{n-i}} \quad \text{for } n \in \mathbb{Z}^+,
\]

by induction on \( n \).
Proof. Prove by induction on $n$. Base case: If $n = 1$, then

$$
\frac{d(uv)}{dx} = u \frac{dv}{dx} + \frac{du}{dx} v = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) u \frac{dv}{dx} + \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \frac{du}{dx} v = \sum_{i=0}^{1} \binom{1}{i} \frac{d^i u}{dx^i} \frac{d^{n-i} v}{dx^{n-i}}.
$$

Inductive step: Suppose that $\frac{d^k(uv)}{dx^k} = \sum_{i=0}^{k} \binom{k}{i} \frac{d^i u}{dx^i} \frac{d^{k-i} v}{dx^{k-i}}$ for some $k \in \mathbb{Z}^+$. Then

$$
\frac{d^{k+1}(uv)}{dx^{k+1}} = \frac{d}{dx} \left( \frac{d^k(uv)}{dx^k} \right) = \frac{d}{dx} \left( \sum_{i=0}^{k} \binom{k}{i} \frac{d^i u}{dx^i} \frac{d^{k-i} v}{dx^{k-i}} \right)
$$

$$
= \sum_{i=0}^{k} \binom{k}{i} \frac{d}{dx} \left( \frac{d^i u}{dx^i} \frac{d^{k-i} v}{dx^{k-i}} \right) = \sum_{i=0}^{k} \binom{k}{i} \left( \frac{d^{i+1} u}{dx^{i+1}} \frac{d^{k-i} v}{dx^{k-i}} + \frac{d^i u}{dx^i} \frac{d^{k-i+1} v}{dx^{k-i+1}} \right)
$$

$$
= \sum_{i=0}^{k} \binom{k}{i} \frac{d^i u}{dx^i} \frac{d^{k-i+1} v}{dx^{k-i+1}} + \sum_{i=0}^{k} \binom{k}{i} \frac{d^{i+1} u}{dx^{i+1}} \frac{d^{k-i} v}{dx^{k-i}}
$$

$$
= \sum_{i=0}^{k} \binom{k}{i} \frac{d^i u}{dx^i} \frac{d^{k-i+1} v}{dx^{k-i+1}} + \sum_{i=1}^{k+1} \binom{k}{i-1} \frac{d^i u}{dx^i} \frac{d^{k-i+1} v}{dx^{k-i+1}}
$$

$$
= \binom{k}{0} \frac{d^{k+1} u}{dx^{k+1}} + \sum_{i=1}^{k} \left[ \binom{k}{i} + \binom{k}{i-1} \right] \frac{d^i u}{dx^i} \frac{d^{k-i+1} v}{dx^{k-i+1}}
$$

$$
= \binom{k}{0} \frac{d^{k+1} u}{dx^{k+1}} + \sum_{i=1}^{k} \binom{k+1}{i} \frac{d^i u}{dx^i} \frac{d^{k-i+1} v}{dx^{k-i+1}} + \binom{k}{k} \frac{d^{k+1} v}{dx^{k+1}}
$$

$$
= \binom{k+1}{0} \frac{d^{k+1} u}{dx^{k+1}} + \sum_{i=1}^{k+1} \binom{k+1}{i} \frac{d^i u}{dx^i} \frac{d^{k+1-i} v}{dx^{k+1-i}}
$$

$$
= \sum_{i=0}^{k+1} \binom{k+1}{i} \frac{d^i u}{dx^i} \frac{d^{k+1-i} v}{dx^{k+1-i}}
$$

$$
= \sum_{i=0}^{k+1} \binom{k+1}{i} \frac{d^i u}{dx^i} \frac{d^{k+1-i} v}{dx^{k+1-i}}
$$
If you prefer prime notation for the derivative:

\[(uv)^{(k+1)} = ((uv)^{(k)})'\]

\[= \left( \sum_{i=0}^{k} \binom{k}{i} u^{(i)} v^{(k-i)} \right)'
\]

\[= \sum_{i=0}^{k} \binom{k}{i} \left( u^{(i)} v^{(k-i)} \right)'
\]

\[= \sum_{i=0}^{k} \binom{k}{i} \left( u^{(i)} v^{(k-i+1)} + u^{(i+1)} v^{(k-i)} \right)
\]

\[= \sum_{i=0}^{k} \binom{k}{i} \left( u^{(i)} v^{(k-i+1)} \right) + \sum_{i=0}^{k} \binom{k}{i} \left( u^{(i+1)} v^{(k-i)} \right)
\]

\[= \sum_{i=0}^{k} \binom{k}{i} \left( u^{(i)} v^{(k-i+1)} \right) + \sum_{i=1}^{k+1} \binom{k}{i-1} \left( u^{(i)} v^{(k-i+1)} \right)
\]

\[= \binom{k}{0} (uv^{(k+1)}) + \sum_{i=1}^{k} \binom{k}{i} \left( u^{(i)} v^{(k-i+1)} \right) + \sum_{i=1}^{k} \binom{k}{i-1} \left( u^{(i)} v^{(k-i+1)} \right)
\]

\[+ \binom{k}{k} \left( u^{(k+1)} v \right)
\]

\[= \binom{k}{0} (uv^{(k+1)}) + \sum_{i=1}^{k} \left[ \binom{k}{i} + \binom{k}{i-1} \right] \left( u^{(i)} v^{(k-i+1)} \right) + \binom{k}{k} \left( u^{(k+1)} v \right)
\]

\[= \binom{k}{0} (uv^{(k+1)}) + \sum_{i=1}^{k} \binom{k+1}{i} \left( u^{(i)} v^{(k-i+1)} \right) + \binom{k}{k} \left( u^{(k+1)} v \right)
\]

\[= \binom{k+1}{0} (uv^{(k+1)}) + \sum_{i=1}^{k} \binom{k+1}{i} \left( u^{(i)} v^{(k-i+1)} \right) + \binom{k+1}{k+1} \left( u^{(k+1)} v \right)
\]

\[= \sum_{i=0}^{k+1} \binom{k+1}{i} \left( u^{(i)} v^{(k+1-i)} \right)
\]

For any nonnegative integer \(n\), show that \(\sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n\).

**Proof.** By the binomial theorem, for any nonnegative integer \(n\), \(3^n = (1 + 2)^n = \)
\[ \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 2^i = \sum_{i=0}^{n} \binom{n}{i} 2^i. \]

- For any positive integer \( n \), prove that \( \sum_{i=0}^{n} \binom{n}{i} = \sum_{i=0}^{n} \binom{n}{i} 2^i \). Hint for one way of proving this: If \( X \) is set of cardinality \( n \), find a bijection between the set of elements of \( \mathcal{P}(X) \) of even cardinality and the set of elements of \( \mathcal{P}(X) \) of odd cardinality.

Proof. Let \( X \) be a nonempty set of cardinality \( n \). Let \( A = \{ S \in \mathcal{P}(X) \mid |S| \text{ is even} \} \) and \( B = \{ S \in \mathcal{P}(X) \mid |S| \text{ is odd} \} \). Fix an element \( x \in X \). Define \( f: A \to B \) by \( f(S) = \begin{cases} S \cup \{ x \} & \text{if } x \not\in S \\ S \setminus \{ x \} & \text{if } x \in S \end{cases} \) for any \( S \in A \). Notice that \( f \) is a map into \( B \) because adding or removing a single element from a set of even cardinality results in a set of odd cardinality.

\( f \) is bijective because \( g: B \to A \) where \( g(S) = \begin{cases} S \setminus \{ x \} & \text{if } x \not\in S \\ S \cup \{ x \} & \text{if } x \in S \end{cases} \) for \( S \in B \) is the inverse of \( f \):

- For \( S \in A \),
  \[
  (g \circ f)(S) = g(f(S)) = \begin{cases} \{ S \cup \{ x \} \} & \text{if } x \not\in S \\ \{ S \setminus \{ x \} \} & \text{if } x \in S \end{cases} = S
  \]

- For \( S \in B \),
  \[
  (f \circ g)(S) = f(g(S)) = \begin{cases} \{ f(S) \cup \{ x \} \} & \text{if } x \not\in S \\ \{ f(S) \setminus \{ x \} \} & \text{if } x \in S \end{cases} = S
  \]
Alternative proof: By the binomial theorem, for any positive integer \( n \),

\[
0 = (1 + (-1))^n = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i}(-1)^i
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} (-1)^i
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} (-1)^i + \sum_{i=0}^{n} \binom{n}{i} (-1)^i
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} \quad \text{if } i \text{ is even} + \sum_{i=0}^{n} \binom{n}{i} \quad \text{if } i \text{ is odd}
\]

and therefore \( \sum_{i=0}^{n} \binom{n}{i} \quad \text{if } i \text{ is even} = \sum_{i=0}^{n} \binom{n}{i} \quad \text{if } i \text{ is odd}. \) \( \square \)