Homework 6 Solutions and Rubric

Point distribution (out of 20 points in total):

- Problem 14.1 – 4 points.
- Problem V.4 – 3 points.
- Last digit of $3^{2019}$ – 3 points.
- Completeness – 10 points. This is for attempting the remaining 8 problems.

See the back of the book for the remaining solutions.

Problems V

1. Prove, for positive integers $n$, that 7 divides $6^n + 1$ if and only if $n$ is odd.

   Proof. Let $n$ be a positive integer. Then, since $6 \equiv -1 \mod 7$,
   
   $7$ divides $6^n + 1 \iff 6^n + 1 \equiv 0 \mod 7$
   $\iff (-1)^n + 1 \equiv 0 \mod 7$
   $\iff (-1)^n = -1$
   $\iff n$ is odd.

4. Suppose that a positive integer is written in decimal notation as $n = a_k a_{k-1} \ldots a_2 a_1 a_0$ where $0 \leq a_i \leq 9$. Prove that $n$ is divisible by 11 if and only if the alternating sum of its digits $a_0 - a_1 + \cdots + (-1)^k a_k$ is divisible by 11.

   Proof. Let $n = a_k a_{k-1} \ldots a_2 a_1 a_0$ where $0 \leq a_i \leq 9$. This means that $n = \sum_{i=0}^{k} a_i 10^i$. Since $10 \equiv -1 \mod 11$,
   
   $11$ divides $n \iff n \equiv 0 \mod 11$
   $\iff \sum_{i=0}^{k} a_i 10^i \equiv 0 \mod 11$
   $\iff \sum_{i=0}^{k} a_i (-1)^i \equiv 0 \mod 11$
   $\iff 11$ divides $\sum_{i=0}^{k} a_i (-1)^i$.

18. Which of the following formulae define a (well-defined) function $f : \mathbb{Q} \rightarrow \mathbb{Q}$?

   i. $f \left( \frac{a}{b} \right) = \frac{a^2}{b^2}$
\[ f \text{ is well-defined. Let } \frac{a}{b} = \frac{c}{d} \text{ where } a, b, c, d \in \mathbb{Z} \text{ and } b, d \neq 0. \text{ Then } \]
\[ ad = bc \Rightarrow a^2d^2 = b^2c^2 \Rightarrow \frac{a^2}{b^2} = \frac{c^2}{d^2} \Rightarrow f\left(\frac{a}{b}\right) = f\left(\frac{c}{d}\right). \]

ii. \[ f\left(\frac{a}{b}\right) = \frac{a^2}{b^3} \]
\[ f \text{ is not well-defined. } \frac{1}{2} = \frac{2}{4} \text{ but } f\left(\frac{1}{2}\right) = \frac{1}{8} \neq \frac{1}{64} = f\left(\frac{2}{4}\right). \]

iii. \[ f\left(\frac{a}{b}\right) = \frac{b}{a} \]
\[ f \text{ is not well-defined. } f\left(\frac{0}{1}\right) \text{ is not defined.} \]

iv. \[ f\left(\frac{a}{b}\right) = a + b \]
\[ f \text{ is not well-defined. } \frac{1}{2} = \frac{2}{4} \text{ but } f\left(\frac{1}{2}\right) = 3 \neq 6 = f\left(\frac{2}{4}\right). \]

v. \[ f\left(\frac{a}{b}\right) = \frac{a - b}{2b} \]
\[ f \text{ is well-defined. Let } \frac{a}{b} = \frac{c}{d} \text{ where } a, b, c, d \in \mathbb{Z} \text{ and } b, d \neq 0. \text{ Then } \]
\[ ad = bc \Rightarrow 2ad = 2bc \\
\quad \Rightarrow 2ad - 2bd = 2bc - 2bd \\
\quad \Rightarrow (a - b)2d = 2b(c - d) \\
\quad \Rightarrow \frac{a - b}{2b} = \frac{c - d}{2d} \\
\quad \Rightarrow f\left(\frac{a}{b}\right) = f\left(\frac{c}{d}\right). \]

• Show that the following sets have the same cardinality.

- \( \mathbb{R} \) and \( \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \)

**Proof.** Define the function \( f : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \) by \( f(x) = \arctan x \) for all \( x \in \mathbb{R} \). Since it has an inverse, \( f^{-1} : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R} \) where \( f^{-1}(x) = \tan x \) for all \( x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \), \( f \) is bijective and therefore \( |\mathbb{R}| = \left|\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right|. \]

- \( \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \) and \( (0, 1) \)
Proof. Define the function \( g: (-\frac{\pi}{2}, \frac{\pi}{2}) \to (0, 1) \) by \( g(x) = \frac{1}{\pi}x + \frac{1}{2} \) for all \( x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \). The function is well-defined because

\[
-\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow -\frac{1}{2} < \frac{1}{\pi}x < \frac{1}{2} \Rightarrow 0 < \frac{1}{\pi}x + \frac{1}{2} < 1.
\]

The inverse of \( g \) is \( g^{-1}: (0, 1) \to (-\frac{\pi}{2}, \frac{\pi}{2}) \) where \( g^{-1}(x) = \pi x - \frac{\pi}{2} \) for all \( x \in (0, 1) \), which is well-defined since

\[
0 < x < 1 \Rightarrow 0 < \pi x < \pi \Rightarrow -\frac{\pi}{2} < \pi x - \frac{\pi}{2} < \frac{\pi}{2},
\]

and

\[
(g \circ g^{-1})(x) = \frac{1}{\pi} \left( \pi x - \frac{\pi}{2} \right) + \frac{1}{2} = x \quad \text{for all } x \in (0, 1)
\]

and

\[
(g^{-1} \circ g)(x) = \pi \left( \frac{1}{\pi} x + \frac{1}{2} \right) - \frac{\pi}{2} = x \quad \text{for all } x \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right).
\]

Since \( g \) has an inverse, \( g \) is bijective and hence \( |(-\frac{\pi}{2}, \frac{\pi}{2})| = |(0, 1)|. \)

\[ \square \]

- \( \mathbb{R} \) and \( (0, 1) \)

Proof. Define the function \( h: \mathbb{R} \to (0, 1) \) by \( h = g \circ f \) (from the previous parts). Since \( f \) and \( g \) are both bijective, \( h \) is a bijection, so \( |\mathbb{R}| = |(0, 1)|. \)

\[ \square \]

- \( \mathbb{R}^2 \) and \( \mathbb{C} \)

Proof. Define the function \( f: \mathbb{R}^2 \to \mathbb{C} \) by \( f(a, b) = a + bi \) for all \( (a, b) \in \mathbb{R}^2 \) where \( a, b \in \mathbb{R} \). The inverse of \( f \) is \( f^{-1}: \mathbb{C} \to \mathbb{R}^2 \) where \( f^{-1}(a + bi) = (a, b) \) for all \( a + bi \in \mathbb{C} \) where \( a, b \in \mathbb{R} \) because

\[
(f \circ f^{-1})(a + bi) = f(a, b) = a + bi \quad \text{for all } a + bi \in \mathbb{C}
\]

and

\[
(f^{-1} \circ f)(a, b) = f^{-1}(a + bi) = (a, b) \quad \text{for all } (a, b) \in \mathbb{R}^2.
\]

Since \( f \) has an inverse, \( f \) is bijective and therefore \( |\mathbb{R}^2| = |\mathbb{C}|. \)

\[ \square \]

- Use modular arithmetic to find the last digit of \( 3^{2019} \).

\[
3^{2019} \equiv (3^2)^{1009} \equiv (-1)^{2019} \equiv -3 \equiv 7 \mod 10 \Rightarrow \text{the last digit of } 3^{2019} \text{ is } 7.
\]