Homework 2 Solutions
See the back of the book for the remaining solutions.

Problems I

11. Prove by contradiction that there does not exist a smallest positive real number.

Proof. For a contradiction, suppose that there exists a smallest positive real number, $x$. Since $x > 0$, $\frac{1}{2}x > 0$. Notice that $\frac{1}{2}x < x$, which is a contradiction because $x$ is the smallest positive real number. Therefore, a smallest positive real number does not exist.

14. Prove Bernoulli’s inequality
\[(1 + x)^n \geq 1 + nx\]
for all non-negative integers $n$ and real numbers $x > -1$.

Proof. Let $x$ be a real number such that $x > -1$. We’ll prove the inequality using induction on $n$: When $n = 0$, $(1 + x)^n = (1 + x)^0 = 1 \geq 1 + 0 = 1 + nx$, so the base case holds. Now assume that the inequality holds for $n = k$ where $k$ is a non-negative integer. Then, using the inductive hypothesis, we have
\[
(1 + x)^{k+1} = (1 + x)^k (1 + x) \\
\geq (1 + kx)(1 + x) \\
= 1 + (k + 1)x + kx^2 \\
\geq 1 + (k + 1)x
\]
so the inequality is true for $n = k + 1$. Therefore, the inequality is true for all non-negative integers $n$.

15. For which non-negative integer values of $n$ is $n! \geq 3^n$?

Proof. The inequality is true for all integers $n \geq 7$. We’ll prove this by induction on $n$: If $n = 7$, then $n! = 7! = 5040 \geq 2187 = 3^7 = 3^n$. Now suppose that $k! \geq 3^k$ where $k$ is an integer such that $k \geq 7$. Then, using the inductive hypothesis and the fact that $k + 1 \geq 8 \geq 3$,
\[
(k + 1)! = (k + 1)k! \\
\geq (k + 1)3^k \\
\geq (3)3^k \\
= 3^{k+1}.
\]
This means that the inequality is true for $n = k + 1$. Therefore, the inequality is true for all integers $n \geq 7$. 

17. For a positive integer \( n \) the number \( a_n \) is defined inductively by

\[
\begin{align*}
  a_1 &= 1, \\
  a_{k+1} &= \frac{6a_k + 5}{a_k + 2} \quad \text{for } k \text{ a positive integer.}
\end{align*}
\]

Prove by induction on \( n \) that, for all positive integers,

(i) \( a_n > 0 \)

**Proof.** When \( n = 1 \), \( a_n = a_1 = 1 > 0 \). Suppose that \( a_k > 0 \) where \( k \) is a positive integer. This implies that \( 6a_k + 5 \) and \( a_k + 2 \) are also positive, so \( a_{k+1} = \frac{6a_k + 5}{a_k + 2} > 0 \).

(ii) \( a_n < 5 \)

**Proof.** When \( n = 1 \), \( a_n = a_1 = 1 < 5 \). Suppose that \( a_k < 5 \) where \( k \) is a positive integer. Then, since \( a_k + 2 > 0 \),

\[
   a_k < 5 \Rightarrow 6a_k + 5 < 5a_k + 10 \Rightarrow 6a_k + 5 < 5(a_k + 2) \Rightarrow \frac{6a_k + 5}{a_k + 2} < 5 \Rightarrow a_{k+1} < 5.
\]

(This proof was obtained by working backwards from the desired inequality \( a_{k+1} < 5 \).)

20. Prove that, for a positive integer \( n \), a \( 2^n \times 2^n \) square grid with any one square removed can be covered using L-shaped tiles of the following shape: □.

**Proof.** When \( n = 1 \), we have a \( 2 \times 2 \) square grid. Removing any one square will result in an L-shape and thus can be covered by a single L-shaped tile. Suppose that any \( 2^k \times 2^k \) square grid with exactly one square removed can be covered using L-shaped tiles, where \( k \) is a positive integer. Now consider a \( 2^{k+1} \times 2^{k+1} \) square grid. Divide the grid into four smaller square grids each of size \( 2^k \times 2^k \) by splitting the square horizontally and vertically. Remove any one square from the \( 2^{k+1} \times 2^{k+1} \) grid. Place a single L-shaped tile at the center of the grid (where the four quadrants meet) so that the tile only covers the three quadrants that did not have a square removed. Now we have four \( 2^k \times 2^k \) square grids, each with exactly one square removed or one square already covered by a tile. By the inductive hypothesis, what remains of each of these four quadrants may each be covered by L-shaped tiles. Therefore, the \( 2^{k+1} \times 2^{k+1} \) grid with one square removed may be covered by L-shaped tiles.

Problems II

3. Prove the absorption laws.

   (i) \( A \cap (A \cup B) = A \)
Proof. (⊆) Let \( x \in A \cap (A \cup B) \). By the definition of intersection, \( x \in A \). (⊇) Let \( x \in A \). Since \( A \subseteq A \cup B \), \( x \in A \cup B \). Therefore, \( x \in A \cap (A \cup B) \). \( \square \)

(ii) \( A \cup (A \cap B) = A \)

Proof. (⊆) Let \( x \in A \cup (A \cap B) \). Then \( x \in A \) or \( x \in A \cap B \).
- If \( x \in A \), then we are done.
- If \( x \in A \cap B \), then \( x \in A \) and \( x \in B \).
In either case, \( x \in A \).
(⊇) Let \( x \in A \). By the definition of union, \( x \in A \cup (A \cap B) \).

8. Given sets \( A, B \in \mathcal{P}(X) \), their symmetric difference is defined by

\[
A \Delta B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B).
\]

Prove that

(i) the symmetric difference is associative (\( (A \Delta B) \Delta C = A \Delta (B \Delta C) \) for all \( A, B, C \in \mathcal{P}(X) \)),

Proof. Let \( A, B, C \in \mathcal{P}(X) \).

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Since \( x \in (A \Delta B) \Delta C \) if and only if \( x \in A \Delta (B \Delta C) \), \( (A \Delta B) \Delta C = A \Delta (B \Delta C) \). \( \square \)

(ii) there exists a unique set \( N \in \mathcal{P}(X) \) such that \( A \Delta N = A \) for all \( A \in \mathcal{P}(X) \)

Proof. Take \( N = \emptyset \). Then for any \( A \in \mathcal{P}(X) \), \( A \Delta N = (A - N) \cup (N - A) = (A - \emptyset) \cup (\emptyset - A) = A \cup \emptyset = A \).

To show uniqueness, assume that \( N' \in \mathcal{P}(X) \) also has the property that for any \( A \in \mathcal{P}(X) \), \( A \Delta N' = A \). In particular, when \( A = N \), \( N \Delta N' = N \). Similarly, \( A \Delta N = A \) for all \( A \in \mathcal{P}(X) \), so if \( A = N' \), then \( N' \Delta N = N' \).

Notice that the symmetric difference is commutative, so it follows that \( N = N \Delta N' = N' \Delta N = N' \). \( \square \)
(iii) for each \( A \in P(X) \), there exists a unique \( A' \in P(X) \) such that \( A \Delta A' = N \)

Proof. Let \( A \in P(X) \) and take \( A' = A \). Then

\[
A \Delta A' = (A - A') \cup (A' - A) = (A - A) \cup (A - A) = \emptyset \cup \emptyset = \emptyset = N.
\]

For uniqueness, suppose that given \( A \in P(X) \), \( A' \) and \( A'' \) in \( P(X) \) have the property that \( A \Delta A' = N \) and \( A \Delta A'' = N \), respectively. By the associativity and commutativity of the symmetric difference,

\[
\begin{align*}
A \Delta A' &= N \\
A'' \Delta (A \Delta A') &= A'' \Delta N \\
(A'' \Delta A) \Delta A' &= A'' \\
(A \Delta A'') \Delta A' &= A'' \\
N \Delta A' &= A'' \\
A' \Delta N &= A'' \\
A' &= A''.
\end{align*}
\]

(iv) for each \( A, B \in P(X) \), there exists a unique set \( C \) such that \( A \Delta C = B \).

Proof. Let \( A, B \in P(X) \). Take \( C = A \Delta B \). Then

\[
A \Delta C = A \Delta (A \Delta B) = (A \Delta A) \Delta B = N \Delta B = B \Delta N = B.
\]

To prove uniqueness, suppose there is another set \( C' \in P(X) \) such that \( A \Delta C' = B \). Then

\[
\begin{align*}
A \Delta C &= A \Delta C' \\
A \Delta (A \Delta C) &= A \Delta (A \Delta C') \\
(A \Delta A) \Delta C &= (A \Delta A) \Delta C' \\
N \Delta C &= N \Delta C' \\
C \Delta N &= C' \Delta N \\
C &= C'.
\end{align*}
\]

\[\square\]