11. Give a proof or a counterexample for each of the following statements:

(i) \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y > 0 \)

**Proof.** True. Let \( x \in \mathbb{R} \). Take \( y = -x + 1 \). Then \( x + y = x + (-x + 1) = 1 > 0 \).

(ii) \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x - y > 0 \)

**Proof.** True. Let \( x \in \mathbb{R} \). Take \( y = x - 1 \). Then \( x - y = x - (x-1) = 1 > 0 \).

(iii) \( \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y > 0 \)

**Proof.** False. Let \( x \in \mathbb{R} \). Take \( y = -x \). Then \( x + y = x + (-x) = 0 \nless 0 \).

(iv) \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy > 0 \)

**Proof.** False. Take \( x = 0 \). Then for all \( y \in \mathbb{R}, xy = 0 \nless 0 \).

(v) \( \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, xy > 0 \)

**Proof.** False. Let \( x \in \mathbb{R} \). Take \( y = 0 \). Then \( xy = 0 \nless 0 \).

(vi) \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy \geq 0 \)

**Proof.** True. Let \( x \in \mathbb{R} \). Take \( y = 0 \). Then \( xy = 0 \geq 0 \).

(vii) \( \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, xy \geq 0 \)

**Proof.** True. Take \( x = 0 \). Then for all \( y \in \mathbb{R}, xy = 0 \geq 0 \).

(viii) \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, (x + y > 0 \text{ or } x + y = 0) \)

**Proof.** True. Let \( x \in \mathbb{R} \). Take \( y = -x \). Then \( x + y = x + (-x) = 0 \).

(ix) \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, (x + y > 0 \text{ and } x + y = 0) \)

**Proof.** False. Take \( x = 0 \). Let \( y \in \mathbb{R} \). Then \( x + y = 0 \) or \( x + y \neq 0 \), which implies that \( x + y \leq 0 \) or \( x + y \neq 0 \). This is equivalent to \( x + y \neq 0 \) or \( x + y \neq 0 \).

(x) \( (\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y > 0) \) and \( (\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 0) \)
Proof. True. The first statement is part (i), which is true. The second statement is also true because for any \( x \in \mathbb{R} \), if \( y = -x \), then \( x + y = 0 \). Therefore, the conjunction of these statements is true.

Problems IV

1. Prove that if an integer \( n \) is the sum of two squares \((n = a^2 + b^2 \text{ for some } a, b \in \mathbb{Z})\) then \( n = 4q \) or \( n = 4q + 1 \) or \( n = 4q + 2 \) for some \( q \in \mathbb{Z} \). Deduce that 1234567 cannot be written as the sum of two squares.

Proof. Suppose \( n = a^2 + b^2 \) for some \( a, b \in \mathbb{Z} \). By exercise 15.5, any perfect square is equal to \( 4q \) or \( 4q + 1 \) for some \( q \in \mathbb{Z} \). This gives three cases for \( a^2, b^2 \):

- If \( a^2 = 4p \) and \( b^2 = 4q \) for some \( p, q \in \mathbb{Z} \), then
  \[ n = a^2 + b^2 = 4p + 4q = 4(p + q). \]

- If \( a^2 = 4p \) and \( b^2 = 4q + 1 \) for some \( p, q \in \mathbb{Z} \), then
  \[ n = a^2 + b^2 = 4p + 4q + 1 = 4(p + q) + 1. \]

We obtain a similar result when \( a^2 = 4p + 1 \) and \( b^2 = 4q \) for some \( p, q \in \mathbb{Z} \).

- If \( a^2 = 4p + 1 \) and \( b^2 = 4q + 1 \) for some \( p, q \in \mathbb{Z} \), then
  \[ n = a^2 + b^2 = 4p + 1 + 4q + 1 = 4(p + q) + 2. \]

Therefore, \( n = 4q \) or \( n = 4q + 1 \) or \( n = 4q + 2 \) for some \( q \in \mathbb{Z} \).

Since \( 1234567 = 4(308641) + 3 \), 1234567 is not the sum of two squares.

2. Let \( a \) be an integer. Prove that \( a^2 \) is divisible by 5 if and only if \( a \) is divisible by 5.

Proof. \((\Rightarrow)\) Suppose \( a^2 \) is divisible by 5. We may express \( a \) in one of the following ways:

- If \( a = 5q \) for some \( q \in \mathbb{Z} \), then 5 divides \( a \).
- If \( a = 5q + 1 \) for some \( q \in \mathbb{Z} \), then
  \[ a^2 = (5q + 1)^2 = 25q^2 + 10q + 1 = 5(5q^2 + 2q) + 1, \]
  which is a contradiction since 5 divides \( a^2 \).
- If \( a = 5q + 2 \) for some \( q \in \mathbb{Z} \), then
  \[ a^2 = (5q + 2)^2 = 25q^2 + 20q + 4 = 5(5q^2 + 4q) + 4, \]
  which is a contradiction since 5 divides \( a^2 \).
• If \( a = 5q + 3 \) for some \( q \in \mathbb{Z} \), then
\[
\begin{align*}
a^2 &= (5q + 3)^2 = 25q^2 + 30q + 9 = 25q^2 + 30q + 5 + 4 = 5(5q^2 + 6q + 1) + 4,
\end{align*}
\]
which is a contradiction since 5 divides \( a^2 \).

• If \( a = 5q + 4 \) for some \( q \in \mathbb{Z} \), then
\[
\begin{align*}
a^2 &= (5q + 4)^2 = 25q^2 + 40q + 16 = 25q^2 + 40q + 15 + 1 = 5(5q^2 + 8q + 3) + 1,
\end{align*}
\]
which is a contradiction since 5 divides \( a^2 \).

Therefore, 5 must divide \( a \).

\((\Leftarrow)\) Assume \( a \) is divisible by 5. Then \( a = 5q \) for some \( q \in \mathbb{Z} \). Then \( a^2 = (5q)^2 = 25q^2 = 5(5q^2) \), so 5 divides \( a^2 \).

6. Use the Euclidean algorithm to find the greatest common divisors of

(i) 165 and 252

\[
\begin{align*}
165 &= 252(0) + 165 \\
252 &= 165(1) + 87 \\
165 &= 87(1) + 78 \\
87 &= 78(1) + 9 \\
78 &= 9(8) + 6 \\
9 &= 6(1) + 3 \\
6 &= 3(2) + 0
\end{align*}
\]
\[
gcd(165, 252) = 3.
\]

(ii) 4284 and 3480

\[
\begin{align*}
4284 &= 3480(1) + 804 \\
3480 &= 804(4) + 264 \\
804 &= 264(3) + 12 \\
264 &= 12(22) + 0
\end{align*}
\]
\[
gcd(4284, 3480) = 12.
\]

7. Let \( u_n \) be the \( n \)th Fibonacci number (for the definition see Definition 5.4.2). Prove that the Euclidean algorithm takes precisely \( n \) steps to prove that \( gcd(u_{n+1}, u_n) = 1 \).
**Proof.** Recall that the Fibonacci sequence is defined as follows:

\[ u_1 = 1, \quad u_2 = 1, \quad u_{n+1} = u_n + u_{n-1} \quad \text{for } n \geq 2. \]

We prove by induction on \( n \). Base case: When \( n = 1 \), the division algorithm tells us that \( u_2 = u_1(1) + 0 \), in one step we’ve show that \( \gcd(u_2, u_1) = 1 \). Inductive step: Now assume that it takes \( k \) steps to prove that \( \gcd(u_{k+1}, u_k) = 1 \). By the division algorithm, there exist unique \( q, r \in \mathbb{Z} \) such that \( u_{k+2} = u_{k+1}q + r \) and \( 0 \leq r < u_{k+1} \). Since \( u_{k+2} = u_{k+1} + u_k \) and \( 0 < u_k < u_{k+1} \), we have \( q = 1 \) and \( r = u_k \). Then \( \gcd(u_{k+2}, u_{k+1}) = \gcd(u_{k+1}, u_k) \), so it will take \( k \) more steps in the Euclidean algorithm to prove that \( \gcd(u_{k+1}, u_k) = 1 \). Therefore, the Euclidean algorithm takes \( k + 1 \) steps in total to prove that \( \gcd(u_{k+2}, u_{k+1}) = 1 \). \( \square \)

10. In each case of Question 6 write the greatest common divisor as an integral linear combination of the two numbers.

(i) 165 and 252

\[
3 = 9 - 6 \\
\quad = 9 - (78 - 9(8)) \\
\quad = 9(9) - 78 \\
\quad = 9(87 - 78) - 78 \\
\quad = 9(87) - 10(78) \\
\quad = 9(87) - 10(165 - 87) \\
\quad = 19(87) - 10(165) \\
\quad = 19(252 - 165) - 10(165) \\
\quad = 19(252) - 29(165)
\]

(ii) 4284 and 3480

\[
12 = 804 - 264(3) \\
\quad = 804 - (3480 - 804(4))(3) \\
\quad = 13(804) - 3(3480) \\
\quad = 13(4284 - 3480) - 3(3480) \\
\quad = 13(4284) - 16(3480)
\]