Math 170A Final

December 13, 2017

- Please put your name, ID number, and sign and date.
- There are 8 problems worth a total of 200 points.
- **You must show your work to receive credit**.

Print Name: ______________________________

Student ID: ______________________________

Seat Number: ____________

Signature and Date: ______________________________

Problem	Score
1	/25
2	/25
3	/25
4	/25
5	/25

Problem	Score
6	/25
7	/25
8	/25
Total	/200

1. (25 pts) Given the following start for a **Matlab** function:

   function [x] = GaussSeidel(n,A,b,y,N)

   that inputs

   - dimension n;
   - $n \times n$ matrix A;
   - $n \times 1$ vector's b and y;
   - number of iterations N;

   complete the function so that it outputs **Gauss-Seidel**'s approximation $\vec{x}^{(N)}$, when starting with initial guess $\vec{x}^{(0)} = y$. Do **not** use Matlab's in-built matrix inverse or backslash; or constant-vector, vector-vector, matrix-vector, matrix-matrix multiplications or additions.

   Remember, Gauss-Seidel: for $i = 1, \ldots, n$,

$$x_i^{(k+1)} = \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right) / a_{ii}.$$

2. (25 pts) Use the **inner-product form** of the Cholesky method to find the **Cholesky factor** R of

$$A = \begin{bmatrix} 16 & 4 & 8 & 8 \\ 4 & 10 & 8 & 4 \\ 8 & 8 & 12 & 10 \\ 8 & 4 & 10 & 12 \end{bmatrix}.$$

You do not have to simplify r_{44}.

Remember, R is the Cholesky factor of A means $r_{ij} = 0$ for $i > j$ and $r_{ii} > 0$ and $A = R^T R$. Also, remember inner-product formulation formulas:

$$r_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} r_{ki}^2}, \qquad r_{ij} = \left(a_{ij} - \sum_{k=1}^{i-1} r_{ki} r_{kj} \right) \Big/ r_{ii}.$$

3. (25 pts) Let A be an $n \times n$ lower triangular matrix. Furthermore, suppose $n > 3$, and $a_{ij} = 0$ for $i > j + 3$. Count **exactly** the number of **additions/subtractions** and the number of **multiplications/divisions** needed to solve $A\vec{x} = \vec{b}$ by **forward substitution**. You **must** take advantage of the banded structure of A and achieve efficient counts.

   Remember: forward substitution

$$x_i = \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j \right) / a_{ii},$$

   and

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

4. (25 pts) Consider the 2×2 linear system

$$\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

where a_{11}, a_{22}, b_1, b_2 are nonzero **real numbers**. Let $\hat{x} = [\hat{x}_1, \hat{x}_2]^T$ denote the result of **forward substitution** on this linear system, performed in an s**-digit rounding** machine with unit roundoff error $0 \leq u < 1$. Show $\hat{x}$ is the solution of a linear system of the form

$$\begin{bmatrix} c_{11} & 0 \\ 0 & c_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

and that $|c_{ii} - a_{ii}| \leq |a_{ii}|(3u + \mathcal{O}(u^2))$, for $i = 1, 2$.

Remember: unit roundoff error u satisfies $\frac{|z - fl(z)|}{|z|} \leq u$.

Also, remember: for $|z| < 1$, $\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$.

5. (25 pts) Prove **SOR**, applied to $A\vec{x} = \vec{b}$, where A has nonzero diagonal elements, will not converge (for every initial guess) when $\omega \geq 2$ or $\omega \leq 0$.

   Remember, SOR:

$$\vec{x}^{(k+1)} = \left(\frac{1}{\omega}D - E\right)^{-1}\left[\left(\frac{1-\omega}{\omega}D + F\right)\vec{x}^{(k)} + \vec{b}\right],$$

   where $A = D - E - F$, and D is the diagonal, $-E$ the strictly lower triangular, and $-F$ the strictly upper triangular portions of A.

   Also, remember: $\det(B) = \prod_{k=1}^{n} \lambda_k$, where λ_k, for $k = 1, \ldots, n$, are the eigenvalues of the $n \times n$ matrix B.

6. (25 pts) Performing operations as if in a 2-**digit rounding** machine, apply the **Power method** to the matrix

$$A = \begin{bmatrix} -68 & -46 \\ 118 & -32 \end{bmatrix},$$

using initial guess $\vec{q}^{(0)}$ a vector of all ones, to calculate s_2.

Remember Power method iterations: $\vec{q}^{(k+1)} = \frac{A\vec{q}^{(k)}}{s_{k+1}}$, where $s_{k+1} = (A\vec{q}^{(k)})_j$ such that $|(A\vec{q}^{(k)})_j| = \max_{1\le i\le n} |(A\vec{q}^{(k)})_i|$.

7. (25 pts) Given the linear system $A\vec{x} = \vec{b}$, with A nonsingular and $\vec{b} \neq 0$, and the **perturbed** linear system $(A + \delta A)\vec{x} = \vec{b}$, with

$$\frac{||\delta A||}{||A||} < \frac{1}{\kappa(A)},$$

where $||\cdot||$ is an induced norm, **prove**

$$\frac{||\vec{\delta x}||}{||\vec{x}||} \leq \frac{\kappa(A)\frac{||\delta A||}{||A||}}{1 - \kappa(A)\frac{||\delta A||}{||A||}}.$$

Remember, induced norm:

$$||A|| = \max_{\vec{y} \neq 0} \frac{||A\vec{y}||}{||\vec{y}||},$$

and condition number: $\kappa(A) = ||A||||A^{-1}||$.

8. Use Gaussian elimination with **complete pivoting** on

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 2 & -4 \\ 1 & 1 & 2 \end{bmatrix}$$

to find permutation matrices P, Q, upper triangular U, and lower triangular L with 1's on the diagonal, satisfying $PAQ = LU$.

Remember: permutation matrices P satisfy $P^{-1} = P^T$.