Math 170A Final
March 21, 2014

- Please put your name, ID number, and sign and date.
- There are 8 problems worth a total of 200 points.
- **You must show your work to receive credit.**

Print Name: ________________________________

Student ID: ________________________________

Signature and Date: _________________________

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1. (25 pts) Given the following header for a Matlab function:

   function [B] = LUfactorization(A,n)

   complete the function so that it outputs the LU factorization (no pivoting) of the $n \times n$ matrix $A$ in the augmented $n \times n$ matrix $B$. Use only basic programming (do not use Matlab’s in-built scalar-vector multiplication, vector-vector addition, or lu function).
2. (25 pts) Write out (but do not solve) the matrix $A$ and vectors $\vec{x}$ and $\vec{b}$ of the linear system of equations $A\vec{x} = \vec{b}$ involved in solving

$$-u''(t) + 2u'(t) + u(t) = \frac{5t}{4}$$

in the interval $[1, 3]$, with boundary conditions $u(1) = 1$, $u(3) = -1$, when subdividing the interval into $m = 5$ equal subintervals. Remember:

$$u''(t) \approx \frac{u(t + h) - 2u(t) + u(t - h)}{h^2}, \quad u'(t) \approx \frac{u(t + h) - u(t - h)}{2h}.$$
3. (25 pts) Consider an \( m \times m \) network of nodes (see the following figure when \( m = 5 \)) with one equation, one unknown at each node:

Suppose the \( i \)th equation is linear and involves only the unknowns associated with the \( i \)th node and the nodes directly connected to it. (for example, the figure’s 13th equation involves \( x_7, x_{12}, x_{13}, x_{14}, x_{19} \)). Find the bandwidth of the matrix in terms of \( m \) (for general \( m \), not just \( m = 5 \)). Be sure to show your work.
4. (25 pts) Consider the linear system $A\vec{x} = \vec{b}$, where $A$ is an $n \times n$ matrix. Suppose $A$ is sparse, with exactly $m_i$ number of nonzero elements in row $i$. Let $m = \sum_{i=1}^{n} m_i$. Count the number of flops, in terms of $m$, involved in one step of Gauss-Seidel, when taking advantage of the sparseness. Remember, Gauss-Seidel going from step $k$ to $k + 1$ uses, for $i = 1, 2, \ldots, n$,

$$x_i^{(k+1)} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}}{a_{ii}}.$$
5. (25 pts) Find the $PA = LU$ factorization, using Gaussian elimination with row partial pivoting, of the matrix

$$A = \begin{bmatrix} 1 & 2 & 6 \\ -4 & 2 & 2 \\ -2 & 4 & 3 \end{bmatrix}.$$ 

Make sure you write out the $P$, $L$, and $U$ matrices.
6. (25 pts) Let \( \vec{x} \) be the solution of the linear system \( A\vec{x} = \vec{b} \) and consider the perturbed linear system \( (A + \delta A)(\vec{x} + \vec{\delta x}) = \vec{b} \). Suppose \( \delta A \) satisfies

\[
\frac{||\delta A||}{||A||} < \frac{1}{\kappa(A)},
\]

where \( ||\cdot|| \) is an induced matrix norm. Derive an upper bound on \( ||\vec{\delta x}||/||\vec{x}|| \) in terms of \( \kappa(A) \) and \( ||\delta A||/||A|| \) and, in addition, label the step where you used the fact that \( ||\delta A||/||A|| < 1/\kappa(A) \).
7. Consider an $s$-digit rounding machine with unit roundoff error $u < 1$. Now consider the $2 \times 2$ linear system
\[
\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},
\]
where the $a_{ij}$'s and $b_i$'s are already nonzero machine numbers. Let $\hat{x}$ denote the result of back substitution, performed in the machine, on this linear system. Show $\hat{x}$ is the solution of a linear system of the form
\[
\begin{bmatrix} c_{11} & c_{12} \\ 0 & c_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},
\]
Show, furthermore, that $|a_{12} - c_{12}| \leq u|a_{12}|.$
8. Find just the $R$ matrix of the $QR$ factorization for

$$A = \begin{bmatrix} 3 & 5 \\ 4 & -10 \\ 0 & -5 \end{bmatrix}.$$