Math 170A Final

March 24, 2017

- Please put your name, ID number, and sign and date.
- There are 8 problems worth a total of 200 points.
- **You must show your work to receive credit**.

Print Name: _____

Student ID: _____

Signature and Date: _____

Problem	Score
1	/25
2	/25
3	/25
4	/25
5	/25

Problem	Score
6	/25
7	/25
8	/25
Total	/200

Formulas:

- $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$
- $(A\vec{x})_i = \sum_{j=1}^{n} a_{ij}x_j$
- A symmetric means $a_{ij} = a_{ji}$, for all i, j
- A symmetric matrix A has semiband width s means $a_{ij} = 0$ for $|i - j| > s$
- The approximations, $\vec{q}^{(k)}$, of Power method always have a 1 for one of its components
- An iterative method with iteration matrix G has errors satisfying $||\vec{e}^{(k+1)}|| \approx \rho(G)||\vec{e}^{(k)}||$, where $\rho(G) = \max_{\{\lambda \text{ eigenvalues of G}\}} |\lambda|$
- $||\vec{x}||_\infty = \max_{1\le i\le n} |x_i|$
- $||A||_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$
- $||A|| = \max_{\vec{x}\neq 0} \frac{||A\vec{x}||}{||\vec{x}||}$, and maximum achieved at the direction of maximum magnification for A
- $\kappa(A) = ||A|| \cdot ||A^{-1}||$
- For $A\vec{x} = \vec{b}$ and $A(\vec{x} + \vec{\delta x}) = \vec{b} + \vec{\delta b}$, $\frac{||\vec{\delta x}||}{||\vec{x}||} \le \kappa(A)\frac{||\vec{\delta b}||}{||\vec{b}||}$
- Unit roundoff error u is the largest relative error that can occur in making a real number into a machine number
- $\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \ldots$
- Q orthogonal means $Q^{-1} = Q^T$
- Rotator for counterclockwise rotation by θ: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$; rotator for clockwise rotation by θ: $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$
- $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$; $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$; $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

1. (25 pts) Let A be a 90×90, symmetric, banded matrix with **semiband width** 13. **Count** exactly the number of **multiplications/divisions** needed in **matrix-vector multiplication**, $A\vec{x}$, for general 90×1 vectors $\vec{x}$. You **must** efficiently use the banded structure and there is **no** need to count additions/subtractions.

2. (25 pts) Let A be an $n \times n$, **symmetric** matrix and let B be the result of A after just **one** step of Gaussian elimination (so B satisfies $b_{i1} = 0$ for all $i = 2, 3, \ldots, n$). Prove $b_{ij} = b_{ji}$, for all $i = 2, 3, \ldots, n$ and $j = 2, 3, \ldots, n$. Be sure to **label** exactly where you use symmetry of A.

3. (25 pts) Find the $PA = LU$ factorization, using Gaussian elimination with **row partial pivoting**, of the matrix

$$A = \begin{bmatrix} 3 & 6 & 24 \\ -12 & 6 & 6 \\ -6 & 12 & 24 \end{bmatrix}.$$

Make sure you **write out** the P, L, and U matrices.

4. (25 pts) Consider an s-digit rounding **machine** with unit roundoff error $u < 1$. Now consider the 2×2 linear system

$$\begin{bmatrix} 1 & 0 \\ a_{21} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

where a_{21} and the b_i's are **real numbers**. Let $\hat{x}$ denote the result of **forward substitution**, performed **in** the machine, on this linear system. Show $\hat{x}$ is the solution of a linear system of the form

$$\begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

Show, **furthermore**, that $|a_{21} - c_{21}| \leq |a_{21}|(3u + \mathcal{O}(u^2))$.

5. (25 pts) Let A be a large $n \times n$, **sparse** matrix with m nonzero values, and let $\vec{r}, \vec{c}, \vec{v}$ be $m \times 1$ vectors in a **coordinate list** representation of A: each nonzero value in A has its row, column, and value recorded as r_k, c_k, v_k, respectively, for some k.

   Given the following header for a **Matlab** function:

   function [q,lambda] = PowerMethod(r,c,v,m,n,N)

   complete the function so that it outputs the approximate **eigenvector** $\vec{q}$ and **eigenvalue** λ after N iterations of the **Power method** when the **inital guess** is a vector of all 1's. You **must** use the coordinate list representation efficiently (in both flops and comparisons). Also use **only** basic programming and do **not** use Matlab's in-built 'max' function.

6. (25 pts) Let A be an $n \times n$ matrix with **real** and **positive** eigenvalues: $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Consider an **iterative method** using $G_\alpha = I - 3\alpha A$ as **iteration matrix**, which is known to have eigenvalues $1 - 3\alpha\lambda_1, 1 - 3\alpha\lambda_2, \ldots, 1 - 3\alpha\lambda_n$. Find α_{opt}, the **optimal** value of α (the value of α for which the iterative method converges the fastest).

7. (25 pts) Let

$$A = \begin{bmatrix} 1/2 & 1 & 3 \\ -2 & 1 & 1 \\ -1 & 2 & 3 \end{bmatrix}, A^{-1} = \begin{bmatrix} -2/7 & -6/7 & 4/7 \\ -10/7 & -9/7 & 13/7 \\ 6/7 & 4/7 & -5/7 \end{bmatrix}.$$

Find $\vec{b}$, $\vec{\delta b}$, and $||\vec{\delta x}||_\infty / ||\vec{x}||_\infty$ where: $A\vec{x} = \vec{b}$, $A(\vec{x} + \vec{\delta x}) = \vec{b} + \vec{\delta b}$, $||\vec{b}||_\infty = 1$, $||\vec{\delta b}||_\infty = 1/2000$, and $||\vec{\delta x}||_\infty / ||\vec{x}||_\infty$ is the **largest** it can be.

8. Let

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}.$$

Find a **rotator** Q and a **lower triangular** matrix L such that $A = QL$.