• Please put your name, ID number, and sign and date.
• There are 8 problems worth a total of 200 points.
• You must show your work to receive credit.

Print Name: ____________________________________________

Student ID: ____________________________________________

Signature and Date: ____________________________________

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Total /200
Formulas:

- \( \sum_{i=1}^{k} i = \frac{k(k+1)}{2} \)
- \((A\vec{x})_i = \sum_{j=1}^{n} a_{ij}x_j\)
- A symmetric means \(a_{ij} = a_{ji}\), for all \(i, j\)
- A symmetric matrix \(A\) has semiband width \(s\) means \(a_{ij} = 0\) for \(|i - j| > s\)
- The approximations, \(\vec{q}^{(k)}\), of Power method always have a 1 for one of its components
- An iterative method with iteration matrix \(G\) has errors satisfying \(||\vec{e}^{(k+1)}|| \approx \rho(G)||\vec{e}^{(k)}||\), where \(\rho(G) = \max\{\lambda\text{ eigenvalues of } G\} |\lambda|\)
- \(||\vec{x}||_\infty = \max_{1 \leq i \leq n} |x_i|\)
- \(||A||_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|\)
- \(||A|| = \max_{\vec{x} \neq 0} \frac{||A\vec{x}||}{||\vec{x}||}, \text{ and maximum achieved at the direction of maximum magnification for } A\)
- \(\kappa(A) = ||A|| \cdot ||A^{-1}||\)
- For \(A\vec{x} = \vec{b}\) and \(A(\vec{x} + \vec{\delta x}) = \vec{b} + \vec{\delta b}\), \(\frac{||\delta x||}{||\vec{x}||} \leq \kappa(A) \frac{||\delta b||}{||\vec{b}||}\)
- Unit roundoff error \(u\) is the largest relative error that can occur in making a real number into a machine number
- \(\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \ldots\)
- \(Q\) orthogonal means \(Q^{-1} = Q^T\)
- Rotator for counterclockwise rotation by \(\theta\): \([\cos \theta \quad -\sin \theta] \quad [\sin \theta \quad \cos \theta]\); rotator for clockwise rotation by \(\theta\): \([\cos \theta \quad \sin \theta] \quad [-\sin \theta \quad \cos \theta]\)
- \(\tan \theta = \frac{\text{opposite}}{\text{adjacent}}; \sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}; \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}\)
1. (25 pts) Let $A$ be a $90 \times 90$, symmetric, banded matrix with semiband width 13. Count exactly the number of multiplications/divisions needed in matrix-vector multiplication, $A\vec{x}$, for general $90 \times 1$ vectors $\vec{x}$. You must efficiently use the banded structure and there is no need to count additions/subtractions.
2. (25 pts) Let $A$ be an $n \times n$, symmetric matrix and let $B$ be the result of $A$ after just one step of Gaussian elimination (so $B$ satisfies $b_{i1} = 0$ for all $i = 2, 3, \ldots, n$). Prove $b_{ij} = b_{ji}$, for all $i = 2, 3, \ldots, n$ and $j = 2, 3, \ldots, n$. Be sure to label exactly where you use symmetry of $A$. 
3. (25 pts) Find the $PA = LU$ factorization, using Gaussian elimination with **row partial pivoting**, of the matrix

$$A = \begin{bmatrix} 3 & 6 & 24 \\ -12 & 6 & 6 \\ -6 & 12 & 24 \end{bmatrix}.$$ 

Make sure you **write out** the $P$, $L$, and $U$ matrices.
4. (25 pts) Consider an $s$-digit rounding machine with unit roundoff error $u < 1$. Now consider the $2 \times 2$ linear system

\[
\begin{bmatrix}
1 & 0 \\
a_{21} & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\
b_2
\end{bmatrix},
\]

where $a_{21}$ and the $b_i$’s are real numbers. Let $\hat{x}$ denote the result of forward substitution, performed in the machine, on this linear system. Show $\hat{x}$ is the solution of a linear system of the form

\[
\begin{bmatrix}
c_{11} & 0 \\
c_{21} & c_{22}
\end{bmatrix}
\begin{bmatrix}
\hat{x}_1 \\
\hat{x}_2
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\
b_2
\end{bmatrix},
\]

Show, furthermore, that $|a_{21} - c_{21}| \leq |a_{21}|(3u + O(u^2))$. 
5. (25 pts) Let $A$ be a large $n \times n$, sparse matrix with $m$ nonzero values, and let $\vec{r}, \vec{c}, \vec{v}$ be $m \times 1$ vectors in a coordinate list representation of $A$: each nonzero value in $A$ has its row, column, and value recorded as $r_k, c_k, v_k$, respectively, for some $k$.

Given the following header for a Matlab function:

```matlab
function [q,lambda] = PowerMethod(r,c,v,m,n,N)
```

complete the function so that it outputs the approximate eigenvector $\vec{q}$ and eigenvalue $\lambda$ after $N$ iterations of the Power method when the initial guess is a vector of all 1’s. You must use the coordinate list representation efficiently (in both flops and comparisons). Also use only basic programming and do not use Matlab’s in-built ‘max’ function.
6. (25 pts) Let $A$ be an $n \times n$ matrix with real and positive eigenvalues: $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Consider an iterative method using $G_{\alpha} = I - 3\alpha A$ as iteration matrix, which is known to have eigenvalues $1 - 3\alpha \lambda_1, 1 - 3\alpha \lambda_2, \ldots, 1 - 3\alpha \lambda_n$. Find $\alpha_{\text{opt}}$, the optimal value of $\alpha$ (the value of $\alpha$ for which the iterative method converges the fastest).
7. (25 pts) Let

\[
A = \begin{bmatrix}
1/2 & 1 & 3 \\
-2 & 1 & 1 \\
-1 & 2 & 3
\end{bmatrix},
A^{-1} = \begin{bmatrix}
-2/7 & -6/7 & 4/7 \\
-10/7 & -9/7 & 13/7 \\
6/7 & 4/7 & -5/7
\end{bmatrix}.
\]

Find $\vec{b}$, $\delta \vec{b}$, and $||\delta \vec{x}||_\infty / ||\vec{x}||_\infty$ where: $A\vec{x} = \vec{b}$, $A(\vec{x} + \delta \vec{x}) = \vec{b} + \delta \vec{b}$, $||\vec{b}||_\infty = 1$, $||\delta \vec{b}||_\infty = 1/2000$, and $||\delta \vec{x}||_\infty / ||\vec{x}||_\infty$ is the largest it can be.
8. Let

\[
A = \begin{bmatrix}
2 & -3 \\
3 & 4
\end{bmatrix}.
\]

Find a rotator \( Q \) and a lower triangular matrix \( L \) such that \( A = QL \).