Homework #2

1.4.15:

(a) Note

\[ A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^2, \]

so \( A = M^T M \), where \( M \) is the diagonal matrix

\[ M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \]

with \( \det M = 6 \neq 0 \), so \( M \) is invertible. Thus \( A \) is positive definite.

(b) With

\[ M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \]

note \( M \) diagonal, and so is upper triangular, with positive diagonal elements. Thus \( R = M \) is the unique Cholesky factor of \( A \).

(c) Note we are looking for upper triangular \( R \) such that \( A = R^T R \), but these \( R \) do not have to have positive diagonal elements. Thus,

\[ \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \]

all are other valid \( R \).

(d) Going through the inner product formulation of Cholesky’s method, we see our first choice is \( r_{11} = \pm a_{11} \), and this choice determines \( r_{1j} \), for \( j = 2, \ldots, n \). Then there is another choice of \( r_{22} = \pm \sqrt{a_{22} - r_{12}^2} \), and this choice determines \( r_{2j} \), for \( j = 3, \ldots, n \). Continuing, this process, we see altogether there are two possible choices for \( r_{11} \), and within these, two possible choices for \( r_{22} \), and within these, two possible choices for \( r_{33} \), and so on, until \( r_{nn} \). Each choice creates a unique \( R \), since it creates a unique vector of elements down the diagonal. The total number of choices thus \( \prod_{i=1}^{n} 2 = 2^n \).

1.4.21:

(a) \( r_{11} = \sqrt{16} = 4 > 0 \); then \( r_{12} = 4/4 = 1 \), and \( r_{13} = 8/4 = 2 \), and \( r_{14} = 4/4 = 1 \); then \( r_{22} = \sqrt{10 - 1} = 3 > 0 \), and \( r_{23} = (8 - 2)/3 = 2 \), and \( r_{24} = (4 - 1)/3 = 1 \); then \( r_{33} = \sqrt{12 - 4 - 4} = 2 > 0 \), and \( r_{34} = (10 - 2 - 2)/2 = 3 \); then \( r_{44} = \sqrt{12 - 1 - 1 - 9} = 1 > 0 \). So the Cholesky factor is

\[ R = \begin{bmatrix} 4 & 1 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

and \( A \) is positive definite.
so $y_1 = 32/4 = 8, y_2 = (26 - 8)/3 = 6, y_3 = (38 - 16 - 12)/2 = 5, y_4 = 30 - 8 - 6 - 15 = 1$. Then

$$
\begin{bmatrix}
4 & 1 & 2 & 1 \\
0 & 3 & 2 & 1 \\
0 & 0 & 2 & 3 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
8 \\
6 \\
5 \\
1 \\
\end{bmatrix},
$$

so $x_4 = 1, x_3 = (5 - 3)/2 = 1, x_2 = (6 - 1 - 2)/3 = 1, x_3 = (8 - 1 - 2 - 1)/4 = 1$.

1.4.35: Partition

$$
A = \begin{bmatrix}
A_j & A_{12} \\
A_{21} & A_{22} \\
\end{bmatrix},
R = \begin{bmatrix}
R_j & R_{12} \\
0 & R_{22} \\
\end{bmatrix},
$$

where $A_j, R_j$ are the $j$th leading principal submatrices of $A$ and $R$, respectively. Then

$$
R^T R = \begin{bmatrix}
R_j^T & 0 \\
R_{12}^T & R_{22}^T \\
\end{bmatrix}
\begin{bmatrix}
R_j & R_{12} \\
0 & R_{22} \\
\end{bmatrix}
= 
\begin{bmatrix}
R_j^T R_j & R_j^T R_{12} \\
R_{12}^T R_j & R_{12}^T R_{12} + R_{22}^T R_{22} \\
\end{bmatrix},
$$

so $A = R^T R$ implies $A_j = R_j^T R_j$. Now, $R$ upper triangular does imply $R_j$ upper triangular, and $R$ has positive diagonal elements implies $R_j$ has positive diagonal elements, since

$$
R = \begin{bmatrix}
R_j & R_{12} \\
0 & R_{22} \\
\end{bmatrix}.
$$

Thus, $R_j$ is the Cholesky factor of $A_j$.

Programming:

(a) See “hw2afn.m”.
(b) See “hw2bfn.m”.
(c) See “hw2cfm.m”.
(d) The first function returns the matrix

$$
\begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \\
\end{bmatrix}.
$$
The second function returns the result of 32 flops. The third function returns the result of 24 flops.