Homework #2

1.3.11: Solve top equation to get $y_1 = 2/2 = 1$; then

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix},$$

then solve top equation to get $y_2 = 4/2 = 2$; then

$$\begin{bmatrix} -1 & 0 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix},$$

then solve top equation to get $y_3 = -3/(-1) = 3$; then

$$3y_4 = 3 - (-3)3 = 12$$

and $y_4 = 4$.

1.4.15:

(a) Note

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^2,$$

so $A = M^T M$, where $M$ is the diagonal matrix

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

with $\det M = 6 \neq 0$, so $M$ is invertible. Thus $A$ is positive definite.

(b) With

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

note $M$ diagonal, and so is upper triangular, with positive diagonal elements. Thus $R = M$ is the unique Cholesky factor of $A$.

(c) Note we are looking for upper triangular $R$ such that $A = R^T R$, but these $R$ do not have to have positive diagonal elements. Thus,

$$\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$

all are other valid $R$.  

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(d) Going through the inner product formulation of Cholesky’s method, we see our first choice is \( r_{11} = \pm a_{11} \), and this choice determines \( r_{1j} \), for \( j = 2, \ldots, n \). Then there is another choice of \( r_{22} = \pm \sqrt{a_{22} - r_{12}^2} \), and this choice determines \( r_{2j} \), for \( j = 3, \ldots, n \). Continuing, this process, we see altogether there are two possible choices for \( r_{11} \), and within these, two possible choices for \( r_{22} \), and within these, two possible choices for \( r_{33} \), and so on, until \( r_{nn} \). Each choice creates a unique \( R \), since it creates a unique vector of elements down the diagonal. The total number of choices thus \( \prod_{i=1}^{n} 2 = 2^n \).

1.4.21:

(a) \( r_{11} = \sqrt{16} = 4 > 0 \); then \( r_{12} = 4/4 = 1 \), and \( r_{13} = 8/4 = 2 \), and \( r_{14} = 4/4 = 1 \); then \( r_{22} = \sqrt{10 - 3} = 3 > 0 \), and \( r_{23} = (8 - 2)/3 = 2 \), and \( r_{24} = (4 - 1)/3 = 1 \); then \( r_{33} = \sqrt{12 - 4 - 4} = 2 > 0 \), and \( r_{34} = (10 - 2 - 2)/2 = 3 \); then \( r_{44} = \sqrt{12 - 1 - 1 - 9} = 1 > 0 \). So the Cholesky factor is

\[
R = \begin{bmatrix}
4 & 1 & 2 & 1 \\
0 & 3 & 2 & 1 \\
0 & 0 & 2 & 3 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

and \( A \) is positive definite.

(b) \[
\begin{bmatrix}
4 & 0 & 0 & 0 \\
1 & 3 & 0 & 0 \\
2 & 2 & 2 & 0 \\
1 & 1 & 3 & 1 \\
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
32 \\
26 \\
38 \\
30 \\
\end{bmatrix},
\]

so \( y_1 = 32/4 = 8, y_2 = (26-8)/3 = 6, y_3 = (38-16-12)/2 = 5, y_4 = 30-8-6-15 = 1 \). Then

\[
\begin{bmatrix}
4 & 1 & 2 & 1 \\
0 & 3 & 2 & 1 \\
0 & 0 & 2 & 3 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
8 \\
6 \\
5 \\
1 \\
\end{bmatrix},
\]

so \( x_4 = 1, x_3 = (5 - 3)/2 = 1, x_2 = (6 - 1 - 2)/3 = 1, x_3 = (8 - 1 - 2 - 1)/4 = 1. \)

1.4.23: The Matlab result for \( A \) is

\[
\begin{bmatrix}
3 & 1 & 1 \\
0 & 3 & 2 \\
0 & 0 & 2 \\
\end{bmatrix},
\]

however, \( A \) is not symmetric, so it is not positive definite. The Matlab result for \( B \) is

\[
\begin{bmatrix}
2 & 1 & 3 \\
0 & 1 & 2 \\
0 & 0 & 4 \\
\end{bmatrix},
\]
which has all positive diagonal entries, and \( B \) is symmetric, so \( B \) is positive definite. The Matlab result for \( C \) states an error using \texttt{“chol”}, since the matrix must be positive definite. Thus, \( C \) is not positive definite. The Matlab result for \( D \) states an error using \texttt{“chol”}, since the matrix must be positive definite. Thus, \( D \) is not positive definite.

**Programming:**

(a) See “hw2afn.m”.

(b) See “hw2bfn.m”.

(c) See “hw2cfn.m”.

(d) The first function returns the matrix

\[
\begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \\
\end{bmatrix}
\]

The second function returns the result of 32 flops. The third function returns the result of 24 flops.