Homework #2

1.4.15:

(a) Note

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^2,$$

so $A = M^T M$, where M is the diagonal matrix

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

with $\det M = 6 \neq 0$, so M is invertible. Thus A is positive definite.

(b) With

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

note M diagonal, and so is upper triangular, with positive diagonal elements. Thus $R = M$ is the unique Cholesky factor of A.

(c) Note we are looking for upper triangular R such that $A = R^T R$, but these R do not have to have positive diagonal elements. Thus,

$$\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$

all are other valid R.

(d) Going through the inner product formulation of Cholesky's method, we see our first choice is $r_{11} = \pm a_{11}$, and this choice determines r_{1j}, for $j = 2, \ldots, n$. Then there is another choice of $r_{22} = \pm\sqrt{a_{22} - r_{12}^2}$, and this choice determines r_{2j}, for $j = 3, \ldots, n$. Continuing, this process, we see altogether there are two possible choices for r_{11}, and within these, two possible choices for r_{22}, and within these, two possible choices for r_{33}, and so on, until r_{nn}. Each choice creates a unique R, since it creates a unique vector of elements down the diagonal. The total number of choices thus $\prod_{i=1}^{n} 2 = 2^n$.

1.4.21:

(a) $r_{11} = \sqrt{16} = 4 > 0$; then $r_{12} = 4/4 = 1$, and $r_{13} = 8/4 = 2$, and $r_{14} = 4/4 = 1$; then $r_{22} = \sqrt{10 - 1} = 3 > 0$, and $r_{23} = (8 - 2)/3 = 2$, and $r_{24} = (4 - 1)/3 = 1$; then $r_{33} = \sqrt{12 - 4 - 4} = 2 > 0$, and $r_{34} = (10 - 2 - 2)/2 = 3$; then $r_{44} = \sqrt{12 - 1 - 1 - 9} = 1 > 0$. So the Cholesky factor is

$$R = \begin{bmatrix} 4 & 1 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and A is positive definite.

(b)

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 32 \\ 26 \\ 38 \\ 30 \end{bmatrix},$$

so $y_1 = 32/4 = 8, y_2 = (26-8)/3 = 6, y_3 = (38-16-12)/2 = 5, y_4 = 30-8-6-15 = 1$. Then

$$\begin{bmatrix} 4 & 1 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 5 \\ 1 \end{bmatrix},$$

so $x_4 = 1, x_3 = (5-3)/2 = 1, x_2 = (6-1-2)/3 = 1, x_3 = (8-1-2-1)/4 = 1$.

1.4.35: Partition

$$A = \begin{bmatrix} A_j & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, R = \begin{bmatrix} R_j & R_{12} \\ 0 & R_{22} \end{bmatrix},$$

where A_j, R_j are the jth leading principal submatrices of A and R, respectively. Then

$$\begin{aligned} R^T R &= \begin{bmatrix} R_j^T & 0 \\ R_{12}^T & R_{22}^T \end{bmatrix} \begin{bmatrix} R_j & R_{12} \\ 0 & R_{22} \end{bmatrix} \\ &= \begin{bmatrix} R_j^T R_j & R_j^T R_{12} \\ R_{12}^T R_j & R_{12}^T R_{12} + R_{22}^T R_{22} \end{bmatrix}, \end{aligned}$$

so $A = R^T R$ implies $A_j = R_j^T R_j$. Now, R upper triangular does imply R_j upper triangular, and R has positive diagonal elements implies R_j has positive diagonal elements, since

$$R = \begin{bmatrix} R_j & R_{12} \\ 0 & R_{22} \end{bmatrix}.$$

Thus, R_j is the Cholesky factor of A_j.

Programming:

(a) See "hw2afn.m".

(b) See "hw2bfn.m".

(c) See "hw2cfn.m".

(d) The first function returns the matrix

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

The second function returns the result of 32 flops. The third function returns the result of 24 flops.