Homework #3

1.4.56: First note

$$B^T = (X^T AX)^T = X^T A^T (X^T)^T = X^T AX = B,$$

since A is symmetric. Now, given $\vec{x} \neq 0$,

$$\vec{x}^T B\vec{x} = \vec{x}^T X^T AX\vec{x} = (X\vec{x})^T A(X\vec{x}).$$

Since X is nonsingular and $\vec{x} \neq 0$, $X\vec{x} \neq 0$. Thus, A positive definite implies $(X\vec{x})^T A(X\vec{x}) > 0$, which implies B is positive definite.

1.5.4: The systems has one equation at each node, which comes out to m^2 equations.

Roughly, the number of nonzeros per row is ≤ 5, since the equation at each node links that node with, at most, its up, down, left, right neighboring nodes. In detail, rows $1, m, m^2 - m + 1, m^2$ have 3 nonzeros; rows $i, m^2 - m + i, 1 + (i-1)m, im$, for all $2 \leq i \leq m-1$, have 4 nonzeros; and the rest of the rows have 5 nonzeros.

The bandwidth of the system is $2m + 1$ since the equation at the kth node, for nodes away from the boundaries of the network of nodes, has nonzeros associated to nodes where the smallest index is $k - m$ and largest is $k + m$. This counts $2m + 1$ columns from $k - m$ to $k + m$, with m of them smaller than k, and m of them larger than k. For nodes at the boundaries of the network, the nonzeros will be in the band defined by the nodes away from the boundaries.

(a) $m = 100$ implies $10,000$ total equations, ≤ 5 nonzeros per row, and a bandwidth of 201.

(b) $m = 1000$ implies $1,000,000$ total equations, ≤ 5 nonzeros per row, and a bandwidth of 2001.

1.5.11:

(a) $m = 100$ implies the semibandwidth is $s = 100$ and $n = 10,000$. Cholesky's method without exploiting band structure has a flop count of roughly $\frac{1}{3}n^3 \approx 3.3333 \cdot 10^{11}$, while Cholesky's method exploiting band structure has a flop count of roughly $ns^2 = 10^8$. Exploiting thus cuts the flop count by a factor of roughly 3333.3.

Without exploiting band structure or symmetry, the storage required for the matrix is $n^2 = 10^8$ real numbers, while exploiting the band structure reduces the storage to $\leq n(s+1) = 1.01 \cdot 10^6$ real number, which is 1.01% of the previous.

(b) Flops without exploitation: $\approx 3.333 \cdot 10^{17}$. Flops with exploitation: 10^{12}. The cut is by a factor of roughly $3.333 \cdot 10^5$.

Storage without exploitation: 10^{12}. Storage with exploitation: $1.001 \cdot 10^9$. This is 0.1001% of the previous.

Programming:

(a) See “hw3afn.m”.

(b) For $n = 10$, the function returns 27 flops. For $n = 100$, the function returns 297 flops. For $n = 400$, the function returns 1197 flops.