Math 170B Final
June 15, 2017

- Please put your name, ID number, circle your section, and sign and date.
- There are 8 problems worth a total of 200 points.
- **You must show your work to receive credit.**

Print Name: ____________________________________________

Student ID: _____________________________________________

Section: A01-Tu 5pm    A02-Tu 6pm    A03-Tu 7pm    A04-Tu 8pm

Signature and Date: ______________________________________

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<th>Problem</th>
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1. (25 pts) Given

- integer \( N > 0 \);
- vector \( x \) of distinct nodes with \( N \) components, ordered from smallest to largest;
- vector \( y \) of nonzero values at nodes with \( N \) components;

suppose the **piecewise linear** interpolant \( P(x) \) of this data has a root. Complete the **Matlab** function with header

```
function [root] = linearinterpolationroot(N,x,y)
```

so that it outputs the **smallest root** of \( P(x) \).

Remember, the piecewise linear interpolant is only defined between the smallest and largest nodes.
2. (25 pts) Consider the fixed point problem with

\[ g(x) = \frac{1}{\sqrt{3 + x^2}}. \]

Use the fixed point theorem to show fixed point iterations using \( g(x) \) converge to a fixed point \( p \in [0, 1] \) for all initial guesses \( p_0 \in [0, 1] \).

Remember, the fixed point theorem: If \( g(x) \) is continuously differentiable in \([a, b]\) and \( g : [a, b] \to [a, b] \) and \( |g'(x)| \leq k < 1 \) for \( x \in [a, b] \), then fixed point iterations using \( g(x) \) converge to a fixed point \( p \in [a, b] \) for all initial guesses \( p_0 \in [a, b] \).
3. (25 pts) Suppose $f(x)$ is twice continuously differentiable for all $x$, and $f''(x) > 0$ for all $x$, and $f(x)$ has a root at $p$ satisfying $f'(p) < 0$. Let $p_n$ be Newton’s method’s sequence of approximations for initial guess $p_0 < p$. Prove $p_1 > p_0$ and $p_1 < p$.

Remember, Newton’s method is $p_{n+1} = p_n - f(p_n)/f'(p_n)$ and

$$p_{n+1} - p = \frac{1}{2} \frac{f''(\xi_n)}{f'(p_n)} (p_n - p)^2.$$ 

for some $\xi_n$ between $p_n$ and $p$. 
4. (25 pts) Let \( x_0, \ldots, x_n \) be distinct nodes in \([a, b]\) and let \( f(x) \) be infinitely continuously differentiable in \([a, b]\). If \( H(x) \) is the **Hermite interpolating polynomial** for

\[
\begin{array}{c|ccc}
  x & x_0 & \ldots & x_n \\
  f(x) & f(x_0) & \ldots & f(x_n) \\
  f'(x) & f'(x_0) & \ldots & f'(x_n) \\
\end{array}
\]

prove, for \( x \in [a, b] \) fixed,

\[
f(x) - H(x) = \frac{f^{(m)}(\xi)}{k!} \prod_{i=0}^{n} (x - x_i)^{q_i},
\]

for some \( \xi \in [a, b] \) and positive integers \( m, k, \) and \( q_i \), for \( i = 0, \ldots, n \)

Hint: **Guess** \( m, k, \) and \( q_i \) first and then study

\[
g(t) = f(t) - H(t) - \frac{\prod_{i=0}^{n} (t - x_i)^{q_i}}{\prod_{i=0}^{n} (x - x_i)^{q_i}} [f(x) - H(x)].
\]

Remember, Generalized Rolle’s Theorem says if \( r_0 \leq r_1 \leq \ldots \leq r_m \) are roots of \( g \in C^m[r_0, r_m] \), then there exists \( \xi \in (r_0, r_m) \) such that \( g^{(m)}(\xi) = 0 \).
5. (25 pts) Find $a, d, B, C$ such that following is a natural cubic spline:

$$S(x) = \begin{cases} 
S_0(x) = 1 + Bx + Cx^2 - 4x^3, & x \in [0, 1] \\
S_1(x) = a - 2(x - 2) + d(x - 2)^3, & x \in [1, 2] 
\end{cases}$$

Remember, natural means $S''$ is zero at the endpoints.
6. (25 pts) Let \( f(x) \) be a function in \([0, 1]\). Derive, from calculus, the normal equations, in matrix form, for its best fitting line minimizing

\[
E(a_0, a_1) = \int_0^1 [2a_0 + a_1(2x - 1) - f(x)]^2 \, dx.
\]

Simplify the matrix, but do not solve the linear system.
7. (25 pts) For \( f(x) \) infinitely continuously differentiable, use **Taylor series** to find \( A, B, \) and \( C \) so that the formula

\[
Af(x - 2h) + Bf(x) + Cf(x + h)
\]
gives the **highest** order accurate approximation of \( f'(x) \) (for general \( f \) and \( x \)). What is that **order**?

Remember, Taylor series says

\[
f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots
\]
8. (25 pts) Consider the table of data

<table>
<thead>
<tr>
<th>$x$</th>
<th>$a$</th>
<th>$a - h$</th>
<th>$a - 3h$</th>
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<tbody>
<tr>
<td>$f(x)$</td>
<td>$f(a)$</td>
<td>$f(a - h)$</td>
<td>$f(a - 3h)$</td>
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Find Newton’s form for the interpolating polynomial, $P(x)$, and simplify $P'(a)$ to the form $Af(a) + Bf(a - h) + Cf(a - 3h)$. What are the constants $A, B, C$?

Remember, Newton’s form is

$$P(x) = f[x_0] + \sum_{i=1}^{n} \left[ f[x_0, \ldots, x_i] \left( \prod_{j=0}^{i-1} (x - x_j) \right) \right]$$

where

$$f[x_k, x_{k+1}, \ldots, x_{i-1}, x_i] = \frac{f[x_{k+1}, \ldots, x_{i-1}, x_i] - f[x_k, x_{k+1}, \ldots, x_{i-1}]}{x_i - x_k}.$$