Math 170B Final

June 12, 2019

- Please put your name, ID number, and sign and date.
- There are 8 problems worth a total of 200 points.
- You must show your work to receive credit.

Print Name: 

Student ID: 

Seat Number: 

Signature and Date: 

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Formulas and Definitions:

- If \( [a, b] = \frac{1}{ad-bc} \)

- Taylor series: \( f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(y)}{k!} (x-y)^k \), if \( f \in C^{n+1}[a, b] \), \( x, y \in [a, b] \), and for some \( \xi \) between \( x, y \).

- Newton’s method: \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \) and \( x_{n+1} - r = \frac{1}{2} \frac{f''(\xi_n)}{f'(x_n)} (x_n - r)^2 \), for some \( \xi_n \) between \( x_n, r \).

- Method of false position is a variation of bisection method where, for each interval \([a, b] \), approximations and interval cut locations are both chosen at \( b_n - \frac{f(b_n)(b_n-a_n)}{f(b_n)-f(a_n)} \).

- Fixed point iterations: \( x_{n+1} = F(x_n) \) and \( x_{n+1} - s = \frac{F'(s)(x_n - s) + \frac{F''(s)}{2!} (x_n - s)^2 + \ldots}{2} \), for \( s \) a fixed point of \( F(x) \).

- If \( \{x_n\}_{n=0}^{\infty} \) converges to \( r \), then it has order of convergence \( q > 0 \) if \( \lim_{n \to \infty} \frac{|x_{n+1} - r|}{|x_n - r|^q} \) exists and is nonzero.

- Divided differences satisfy \( f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0} \).

- If \( x_0, \ldots, x_n \in [a, b] \), then \( f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!} \), for some \( \xi \in (a, b) \).

- Newton form:
  \[
  p(x) = f[x_0] + \sum_{k=1}^{n} \frac{f[x_0, x_1, \ldots, x_k]}{k!} \prod_{j=0}^{k-1} (x - x_j) .
  \]

- The Chebyshev polynomial of degree \( n \), \( T_n(x) \), satisfies: \( T_0(x) = \cos(x) \) and \( T_1(x) = 1 \), \( T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \); and \( |T_{n}(x)| \leq 1 \) in \([-1,1]\), with \( T_n \left( \cos \frac{\pi}{n} \right) = (-1)^j \), for \( j = 0, \ldots, n \); and \( T_n \left( \cos \frac{2j-1}{2n} \pi \right) = 0 \), for \( j = 1, \ldots, n \).

- If \( p(x) \) is a monic polynomial of degree \( n \), then \( \max_{x \in [-1,1]} |p(x)| \geq \max_{x \in [-1,1]} \left| \frac{T_n(x)}{2^{n+1}} \right| \), for \( n \geq 1 \).

- Degree \( \leq d \) spline interpolants, with knots \( t_0 < \ldots < t_n \), are interpolating functions in \( C^k[t_0, t_n] \) that are composed of degree \( \leq d \) polynomials in \([t_i, t_{i+1}]\), with \( k \) as large as possible for general data and \( n \).

- A natural spline satisfies \( S''(t_0) = S''(t_n) = 0 \).

- If \( E \) is a normed vector space, with norm \( \| \cdot \| \), and \( G \) is a subspace of \( E \), the best approximation \( g \in G \) of an \( f \in E \) in \( G \) is defined to satisfy \( \| f - g \| \leq \| f - h \| \), for all \( h \in G \).

- If \( E \) is an inner product space, with inner product \( \cdot \cdot \cdot \cdot \), and \( G \) is a subspace of dimension \( n \), the best approximation \( g \in G \) of an \( f \in E \) in \( G \) uses the norm \( \| h \| = \sqrt{\langle h, h \rangle} \), and satisfies the normal equations \( A^T c = b \), where \( g = \sum_{i=1}^{n} c_i g_i \), \( a_{ij} = \langle g_i, g_j \rangle \), and \( b_i = \langle f, g_i \rangle \), for any basis \( g_1, \ldots, g_n \) of \( G \).

- \( f \perp G \) means \( \langle f, h \rangle = 0 \) for all \( h \in G \).

- \( ||f + g||^2 = ||f||^2 + 2 < f, g > + ||g||^2 \).

- A two-dimensional interpolation polynomial coming from the tensor product method is a linear combination of \( u_i(x)v_j(y) \), where \( u_i, v_j \) are one-dimensional cardinal functions.

Do not remove this page. You can use below and back for scratch work.
1. (25 pts) Given the following header for a Matlab function:

```matlab
function [p] = NewtonForm(m,x,y,z)
```

that inputs

- number of data points \( m \);
- \( m \)-component vectors \( x \) and \( y \), the \( x \)- and \( y \)-coordinates, respectively, of the data points;
- location \( z \);

and uses divided difference tables and **Newton form** to output the value of the interpolation polynomial, for the given data points, at \( z \).
2. (25 pts) Let $f \in C^2[r, b]$, where $r$ is a root of $f$ and $r < b$. Suppose, further, that $f''(x) < 0$ for all $x \in [r, b]$, and $f'(r) < 0$. Prove Newton’s method’s sequence of approximations converges to $r$ for all initial guesses $x_0 \in [r, b]$. 
3. (25 pts)

(a) Find approximations of the form $\frac{p}{q}$, for $p, q$ integers with $q \neq 0$, for $\sqrt{5}$ using $c_0, c_1$ of the method of false position with initial interval $[a_0, b_0]$, where $a_0 < b_0$ are the two integers closest to $\sqrt{5}$.

(b) Let $F(x) = \frac{1}{2x} - \frac{1}{8x^2}$, which has fixed point at $s = \frac{1}{2}$, and let $\{x_n\}_{n=0}^{\infty}$ denote a sequence of fixed point iterations, using fixed point function $F(x)$, satisfying $x_n \neq s$ for all $n \geq 0$, that converges to $s$. Determine the order of convergence of the sequence.
4. (25 pts) Given evenly spaced nodes

\[ 0 = x_0 < x_1 < \ldots < x_{10} = 1 \]

and the values of \( f(x_i) = (6.1)x_i - 1.8 \), \( f'(x_i) = 27x_i - 6 \), and \( f''(x_i) = 60x_i \), for all \( i = 0, \ldots, 10 \), let \( p(x) \) be the piecewise quintic (degree \( \leq 5 \)) interpolant for the data (each piece of polynomial only uses two nodes). Find \( p(0.401) \).
5. (25 pts)

(a) Find $a, b, c, d$ (they exist) such that $S(x)$ is a natural cubic spline interpolant on $[0, 3]$, where

$$S(x) = \begin{cases} 
2a + x - x^3, & x \in [0, 1], \\
1 - b(x - 1) - 3(x - 1)^2 + 4(x - 1)^3, & x \in [1, 2], \\
4(x - 2) - c(x - 2)^2 - 2d(x - 2)^3, & x \in [2, 3].
\end{cases}$$

(b) Find the degree $\leq 2$ polynomial that best approximates

$$f(x) = -5x^3 + 3x^2 - x + 2$$

under the norm $\|g\| = \max_{x \in [-1,1]} |g(x)|$. 
6. (25 pts)

(a) Find the line \( c_0 + c_1 x \) that best approximates \( f(x) = 5x^3 - 3x \) under the norm 
\[ ||g|| = \left( \int_0^1 [g(x)]^2 \, dx \right)^{1/2}. \]

(b) Consider nodes \((x_i, y_j)\), for \( i = 1, 2, 3 \) and \( j = 1, 2 \), where \( x_1 = 0, x_2 = 1, x_3 = 3 \) and \( y_1 = 0, y_2 = 2 \). Given values \( f(x_i, y_j) = x_i^2 + x_i(1 + y_j) \), for \( i = 1, 2, 3 \) and \( j = 1, 2 \), let \( p(x, y) \) denote the interpolation polynomial for the data coming from the tensor product method. Find \( p(2, 1/3) \).
7. (25 pts) Let $x_0 < x_1 < \ldots < x_{2n-1}$ be nodes and $t_0 < t_1 < \ldots < t_n$ be knots, satisfying $x_{2i}, x_{2i+1} \in [t_i, t_{i+1}]$, for all $i = 0, \ldots, n - 1$. Consider a degree $\leq d$ spline $S$ interpolating given data points $(x_i, f(x_i))$, for $i = 0, \ldots, 2n - 1$, with knots at $t_i$, for $i = 0, \ldots, n$. Enforcing the number of unknowns to be always greater than or equal to the number of equations, find the largest $k$ we can choose, when $d = 5$, so that $S \in C^k[t_0, t_n]$, for any $n \geq 1$?
8. (25 pts) Let $G$ be a subspace (not necessarily finite dimensional) in an inner product space $E$. For $f \in E$ and $g \in G$, prove $g$ is a best approximation to $f$ in $G$ if and only if $f - g \perp G$.

Hint: Study $\|f - g + \lambda h\|^2$, for $\lambda > 0$, $h \in G$. 