Math 170B Final
March 16, 2016

- Please put your name, ID number, and sign and date.
- There are 8 problems worth a total of 200 points.
- **You must show your work to receive credit.**

Print Name: ________________________________

Student ID: ________________________________

Signature and Date: _________________________

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1. (25 pts) Write a Matlab program that takes as input

- $m$, the number of nodes;
- $x$, a vector of the data point nodes;
- $y$, a vector of data point height values;
- $z$, a location;

and returns the value of the interpolation polynomial at $z$ using **Newton’s form**.
2. (25 pts) Given the data points \((x_0, f(x_0)), (x_0 + h, f(x_0 + h)), (x_0 + 2h, f(x_0 + 2h))\), for \(h \neq 0\), find the \textbf{Lagrange form} of the interpolation polynomial \(p(x)\) and use it to derive a formula for approximating the value of \(f'(x_0)\). Please simplify the formula.
3. (25 pts)

(a) Let \( f(x) = x^2 + 2x \) and let \( p(x) \) be the **piecewise linear** interpolant for the data points

\[
(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))
\]

where \( x_i = -1 + \frac{2i}{n} \). Find the value of \( p'(0.102) \) when \( n = 100 \).

(b) Suppose \( x_n \) converges to \( r \) with order of convergence \( (1 + \sqrt{5})/2 \). Consider the sequence \( z_n \) defined from \( z_n = x_{2n} \), which we know also converges to \( r \). Find the **order of convergence** of \( z_n \).
(c) Let

\[ S(x) = \begin{cases} 
  S_0(x) = x^3, & \text{if } 0 \leq x < 1 \\
  S_1(x) = a + b(x - 1) + c(x - 1)^2 + d(x - 1)^3, & \text{if } 1 \leq x \leq 3 
\end{cases} \]

Find \( a, b, c, d \) such that \( S(x) \) is a natural cubic spline.

(d) Given the first two approximations \( c_0 = 1.3, c_1 = 0.9 \) of bisection method, determine the starting interval \([a_0, b_0]\) that was used.
4. (25 pts) Let \( f(x) = \sin(\pi x) \) and let \( H(x) \) be the **piecewise cubic** Hermite interpolant for the data:

\[
\begin{array}{c|c|c|c}
  x & x_0 & \ldots & x_n \\
  f(x) & f(x_0) & \ldots & f(x_n) \\
  f'(x) & f'(x_0) & \ldots & f'(x_n)
\end{array}
\]

where \( x_i = -1 + \frac{2i}{n} \), for \( i = 0, 1, \ldots, n \). Using bounds on the error for a Hermite interpolation polynomial \( p(x) \):

\[
 f(x) - p(x) = \frac{f^{(2m+2)}}{(2m + 2)!} \prod_{i=0}^{m} (x - x_i)^2,
\]

find \( n \) such that \( |f(x) - H(x)| \leq 10^{-12} \) will be satisfied.
5. (25 pts) Let
\[ f(x) = \frac{x}{10} + \frac{\sqrt{5}x}{\sqrt{x^2 + 4}} - 1, \]
which has a root \( r \).

(a) Using Newton’s method, we already know the result:
\[ e_{n+1} = \frac{1}{2} \frac{f''(\xi_n)}{f'(x_n)} e_n, \]
where \( e_n = x_n - r \) and for some \( \xi_n \) between \( x_n \) and \( r \). Find an example of \( \delta \) where there exists \( \lambda < 1 \) such that \( |e_{n+1}| \leq \lambda |e_n| \) for all \( |e_n| \leq \delta \).

(b) Use bisection method with starting interval \([0, 1]\) to get an approximation, \( c_n \), that, when used as an initial guess \( x_0 \) for Newton’s method, satisfies \( |e_0| \leq \delta \).
(Hint: evaluating \( f(x) \) is easier with \( f(x) = \frac{x}{10} + \frac{\sqrt{5x\sqrt{x^2 + 4}}}{x^2 + 4} - 1 \))
6. (25 pts) Consider the inner product $< g, h > = \int_a^b g(x) h(x) w(x) \, dx$, where $a = -1, b = 1, w(x) = 1 - x$. Let $f(x) = x^2$.

   (a) Use minimization to derive, in matrix form, the normal equations for the line, of the form $a_0(1 - x) + a_1(1 + x)$, that best approximates $f(x)$ with respect to the norm induced by the inner product: $||g|| = \sqrt{< g, g >}$.

   (b) Solve for $a_0$ and $a_1$ and write the best approximating line in the form $Mx + B$, for some constants $M, B$. 


7. (25 pts) Prove the natural cubic spline $S(x)$ for the data points

$$(x_0, y_0), (x_1, y_1), (x_2, y_2), \text{ with } a = x_0 < x_1 < x_2 = b,$$

satisfies

$$\int_a^b [S''(x)]^2 \, dx \leq \int_a^b [f''(x)]^2 \, dx,$$

for any twice continuously differentiable $f(x)$ in $[a, b]$ that interpolates the data points. (Hint: Set $e(x) = f(x) - S(x)$ and start manipulating $\int_a^b [f''(x)]^2 \, dx$ and use the fact that $S(x)$ is piecewise cubic)
8. (25 pts) Let $f(x)$ be infinitely continuously differentiable in $[a, b]$ and let $p(x)$ be the (general) Hermite interpolation polynomial for the data:

<table>
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<td>$f(x)$</td>
<td>$f(x_0)$</td>
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<td>$f'(x)$</td>
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where $x \in [a, b]$ and $a = x_0 < x_1 < x_2 = b$. Prove the error formula:

$$f(x) - p(x) = \frac{f^{(m)}(\xi_x)}{k!} (x - x_0)^p (x - x_1)^q (x - x_2)^r$$

for some $\xi_x \in [a, b]$ and positive integers $m, k, p, q, r$. (Hint: Correctly guess values of $m, k, p, q, r$ and then set $\phi(t) = f(t) - p(t) - \lambda w(t)$, where $w(t) = (t - x_0)^p (t - x_1)^q (t - x_2)^r$ and $\lambda$ is chosen appropriately)