Homework #1

1. Can the bisection method be safely used to find a root of the following functions using the following intervals? Why or why not?
   (a) \( f(x) = \cos x + e^x \) with \([0, \pi/2]\).
   (b) \( f(x) = x^3 + x + 1 \) with \([-1, 0]\).
   (c) \( f(x) = \frac{1}{x} \) with \([-1, 7]\).
   (d) \( f(x) = \begin{cases} \ -x - 1, & x \leq 0 \\ \ x - 1, & x > 0 \end{cases} \) with \([-2, 1/2]\).

2. Consider \( f(x) = x(x-1)(x+2) \), which has roots at \( x = 0, 1, -2 \). Determine which root the bisection method approximates when using the starting interval \([-3, 2]\).

3. Suppose \( f(x) \) is a given continuous function in \([-1, 4]\) such that \( f(-1) \) and \( f(4) \) have different signs and consider the bisection method on \( f(x) \) using starting interval \([-1, 4]\).
   (a) Bound the absolute error for the approximation \( c_{30}\).
   (b) Use the bound on absolute error to determine which \( c_n \) are guaranteed to have absolute error less than \( 10^{-11} \).

4. Suppose “\( f \) has different sign at \( a \) and \( b \)” is defined to be true if and only if \( f(a)f(b) \leq 0 \).
   (a) Find \( c_n \) generated by the bisection method on \( f(x) = x^2 \) when the starting interval is \([0, b]\), for \( b > 0 \).
   (b) What does this sequence of approximations converge to and what is the order of convergence?

5. (a) Given the first two approximations, \( c_0, c_1 \), generated from a bisection method, determine the starting interval \([a_0, b_0]\) that was used.
   (b) Apply this to find the starting interval when \( c_0 = -0.2 \) and \( c_1 = 0.3 \)

6. Suppose we modify the bisection method into these three variations:
   - Variation #1: Approximations are chosen at the midpoint of the interval. The interval is cut into two at the location \((2a + b)/3\).
   - Variation #2: Approximations are chosen at the location \((2a + b)/3\). The interval is cut into two at the midpoint of the interval.
   - Variation #3: Approximations are chosen at the location \((2a + b)/3\). The interval is cut into two at the location \((2a + b)/3\).

Answer the following questions about these variations:
   (a) Bound the absolute errors of the approximation \( c_n \) for each variation when the starting interval is \([a, b]\). Which variation has the best bound?
(b) Calculate the first 2 approximations \( c_0, c_1 \) for each variation when \( f(x) = \cos x - x \) with starting interval \([0, \pi/2]\).

7. (a) Use the bisection method to generate the approximations \( c_0, c_1, c_2, c_3 \) of \( 2\sqrt{2} \) by finding the positive root of \( x^2 - 8 \) using the starting interval \([2, 3]\).

(b) Find the bound of the absolute error for the final approximation and verify that the actual absolute error satisfies this bound.

(c) Use Newton’s method to generate \( x_1, x_2, x_3 \) starting with \( x_0 = 3 \). Compare the absolute error of \( x_3 \) to that of \( c_3 \).

8. Answer True or False for the following questions (you do not need to show work but write down your explanation if you are unsure). In each, \( f : \mathbb{R} \to \mathbb{R} \) denotes a continuous function with different signs at \( a \) and \( b \).

(a) The bisection method on \( f(x) \) using starting interval \([a, b]\) will always generate a sequence of approximations converging to a root of \( f(x) \).

(b) The bisection method only works when \( f(x) \) has exactly one root in the starting interval \([a, b]\).

(c) When there are multiple roots of \( f(x) \) in the starting interval \([a, b]\), the bisection method will approximate the smallest root.

(d) When there are multiple roots of \( f(x) \) in the starting interval \([a, b]\), the bisection method will approximate the root closest to \((a + b)/2\).

(e) The sequence of approximations generated by the bisection method on \( f(x) \) using starting interval \([a, b]\) is the same as the sequence on \( g(f(x)) \) using \([a, b]\) when \( g : \mathbb{R} \to \mathbb{R} \) is continuous with \( g(x)g(y) < 0 \) for all \( x < 0 \) and \( y > 0 \).

(f) The bisection method on \( g(x) \) using starting interval \([a, b]\) may breakdown if \( g \) is a function that is continuous in \([a, b]\) but not defined outside of \([a, b]\).

9. (Matlab) Given \( f(x) \), suppose “hw1f.m” is a function that takes as input the number \( x \) and outputs \( f(x) \).

(a) Write a Matlab function that inputs:
   - \( a, b \): endpoints of starting interval \([a, b]\);
   - \( n \): number of iterations to perform;

and outputs \( c_n \) using Variation #1 in problem 6 on \( f(x) \). Write or print out your function and turn it in.

(b) Apply your function to \( f(x) = \sin x - 1/2 \) with starting interval \([0, \pi/2]\) for 20 iterations. Write or print out the resulting approximation and compare it to \( \arcsin(1/2) = \pi/6 \).