Homework #2

1. Let \( g \in C^1[a, b] \), for \( a < b \), and suppose \( g(x) \in [a, b] \) for all \( x \in [a, b] \), and \( g'(x) \leq k < 1 \) for all \( x \in [a, b] \). Prove \( g \) cannot have more than one fixed point in \([a, b]\).

2. Let \( g(x) = 1/x + x/2 \) and consider the interval \([1.4, 1.45]\).
   (a) Show \( g(x) \in [1.4, 1.45] \) for \( x \in [1.4, 1.45] \) by finding the maximum value, \( M \), and minimum value, \( m \), of \( g \) in the interval and showing \([m, M] \subseteq [1.4, 1.45]\).
   (b) Find \( k < 1 \) such that \( |g'(x)| \leq k \) for all \( x \in [1.4, 1.45] \) by finding the maximum value of \( |g'(x)| \).
   (c) Use the \( k \) you just found, and the error bound
      \[ |p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}, \]
      to estimate the \( n \) such that \( p_n \) of fixed point iterations will have absolute error
      \( \leq 10^{-8} \) when \( p_0 = 1.425 \).
   (d) Do the same using the error bound
      \[ |p_n - p| \leq \frac{k^n}{1-k} |p_1 - p_0|, \]
      when \( p_0 = 1.425 \).

3. Suppose \( g \in C^1(-\infty, \infty) \) and \( g \) has a fixed point \( p \), and \( |g'(p)| > 1 \). Let \( \{p_n\}_{n=0}^{\infty} \) with no \( p_n \) equal to \( p \), be the sequence of approximations generated by fixed point iterations on \( g \) with initial guess \( p_0 \). Prove there exists \( \delta > 0 \) such that whenever someone chooses an \( N > 0 \), you can then find an \( M \geq N \) such that \( |p_M - p| > \delta \).

4. Let \( f(x) = x^3 - 2x + 1 \). To solve \( f(x) = 0 \), consider the three fixed point functions
   \[
   g_1(x) = \frac{1}{2}(x^3 + 1) \\
   g_2(x) = \frac{2}{x} - \frac{1}{x^2} \\
   g_3(x) = \sqrt{2 - \frac{1}{x}}.
   \]
   (a) Verify \( p = 1 \) is a root of \( f \), and also a fixed point for each of the three fixed point functions.
   (b) Compute \( g_1'(1), g_2'(1), g_3'(1) \) and comment on the convergence of fixed point iteration’s sequence of approximations on each fixed point function. Rank those that converge for sufficiently close initial guesses \( p_0 \) from fastest to slowest convergence.

5. Consider the problem of finding the point on the graph of \( y = x^3 \) closest to the point \((3, -1)\).
(a) Write down the expression for $d(x) = \text{the square of the distance from } (x, x^3) \text{ to } (3, -1)$.

(b) Minimize $d(x)$ by finding the, in this case, unique critical point: approximating the solution of $f(x) = d'(x) = 0$ using Newton’s method to generate $p_3$ when $p_0 = 2$.

6. Consider the equation $xy^2 + \tan y = x$ that implicitly defines $y$ as a function of $x$: $y = y(x)$. Approximate $y(1)$ with $p_3$ generated by Newton’s method using initial guess $p_0 = 1$.

7. Let

$$f(x) = \begin{cases} \sqrt{x}, & \text{if } x \geq 0 \\ -\sqrt{-x}, & \text{if } x < 0. \end{cases}$$

(a) Generate Newton’s method’s $p_1, p_2, p_3$ in terms of $p_0 > 0$. Then guess the value of $p_n$ for $n$ even and for $n$ odd.

(b) Will Newton’s method converge for any $p_0 \neq 0$? Why does this not violate the theorem on the convergence of Newton’s method (Theorem 2.6 on page 69) since the initial guess can be arbitrarily close to the root?

8. Let $f \in C^2[a, p]$, where $p$ is a root of $f$ and $a < p$. Furthermore, suppose $f''(x) < 0$ in $[a, p]$ and $f'(p) > 0$. Prove Newton’s method’s sequence of approximations converges to $p$ for all initial guesses $p_0 \in [a, p]$.

9. (Matlab) Suppose we have two Matlab functions, in “hw2f.m” and “hw2fprime.m”, that both take as input $x$ and output expressions for $f(x)$ and $f'(x)$, respectively. Then write a Matlab function that inputs

- initial guess $p_0$;
- number of iterations $N \geq 1$;

and outputs the $p_N$ of Newton’s method. Make sure you call “hw2f” and “hw2fprime” when you need values of $f(x)$ or $f'(x)$.

(a) Print out or write out your functions, including “hw2f.m” and “hw2fprime.m”, for Newton’s method on $f(x) = x^2 - 8$.

(b) Write out or print out your results when $p_0 = 2$ and $N = 2, 3, 4$. Also write out or print out the absolute errors for each of these approximations (note exact root is $2\sqrt{2}$).

(c) Write out or print out your results with, instead, $f(x) = x^2$, when $p_0 = 1$ and $N = 20, 21, 22$. Also write out or print out the absolute errors for each of these approximations (note exact root is 0).