Homework #2

1. Let \( g(x) = 1/x + x/2 \) and consider the interval \([1.4, 1.45]\).
   
   (a) Show \( g(x) \in [1.4, 1.45] \) for \( x \in [1.4, 1.45] \) by finding the maximum value, \( M \), and minimum value, \( m \), of \( g \) in the interval and showing \([m, M] \subseteq [1.4, 1.45]\).
   
   (b) Find \( 0 \leq \lambda < 1 \) such that \( |g'(x)| \leq \lambda \) for all \( x \in [1.4, 1.45] \) by finding the maximum value of \( |g'(x)| \).
   
   (c) Use the \( \lambda \) you just found to estimate the \( n \) such that \( p_n \) of fixed point iterations will have absolute error \( \leq 10^{-8} \) when \( p_0 = 1.425 \).
   
   (d) Compute \( p_3 \) of the fixed point iteration using \( p_0 = 1.425 \) and find the absolute error of this approximation (exact solution is \( \sqrt{2} \)).

2. Let \( p \) be a fixed point of \( g \). Show if \( |g'(p)| < 1 \), then fixed point iterations will converge if \( p_0 \) is close enough to \( p \) (use continuity of \( g' \) to show there is an interval \([p - \delta, p + \delta]\) where \( |g'(x)| \leq \lambda < 1 \), and then try to use the theorem on convergence of fixed point iterations).

3. Show if there exists a \( \delta > 0 \) such that \( |g'(x)| \geq \lambda > 1 \) in \([p - \delta, p + \delta]\) then a fixed point iterations sequence of approximations \( \{p_n\}_{n=0}^{\infty} \), with \( p_n \neq p \) for all \( n \), will not converge to \( p \).

4. Consider the problem of finding the point on the graph of \( y = x^3 \) closest to the point \((3, -1)\).
   
   (a) Write down the expression for \( d(x) = \) the square of the distance from \((x, x^3)\) to \((3, -1)\).
   
   (b) Minimize \( d(x) \) by finding the, in this case, unique critical point: approximating the solution of \( f(x) = d'(x) = 0 \) using Newton’s method to generate \( p_2 \) when \( p_0 = 2 \).

5. Consider the equation \( xy^2 + \tan y = x \) that implicitly defines \( y \) as a function of \( x \): \( y = y(x) \). Approximate \( y(1) \) with \( y_2 \) generated by 2 iterations of Newton’s method using initial guess \( y_0 = 1 \).

6. Let \( f(x) = \begin{cases} \sqrt{x}, & \text{if } x \geq 0 \\ -\sqrt{-x}, & \text{if } x < 0 \end{cases} \)

   (a) Generate Newton’s method’s \( p_1, p_2, p_3 \) in terms of \( p_0 > 0 \). Then guess the value of \( p_n \) for \( n \) even and \( n \) odd.

   (b) Will Newton’s method converge for any \( p_0 \neq 0 \)? Why does this not violate the theorem on the convergence of Newton’s method since the initial guess can be arbitrarily close to the root?
7. Suppose $f(x)$ is twice continuously differentiable and suppose $f''(x) > 0$ for all $x$. Furthermore, let $p$ be a root of $f(x)$ and suppose $f'(p) > 0$.

(a) Prove Newton’s method’s sequence of approximations $p_n$ satisfy $p_n > p$, when $p_0 > p$.

(b) Prove Newton’s method’s sequence of approximations $p_n$ satisfy $p_{n+1} < p_n$ when $p_0 > p$.

8. (Matlab) First write two Matlab functions that both take as input $x$ and output expressions for $f(x)$ and $f'(x)$ (calculate expression for $f'(x)$ by hand). Then write a Matlab function that inputs

- initial guess $p_0$;
- number of iterations $N$;

and outputs the $p_N$ of Newton’s method. Make sure you call the functions you have when you need values of $f(x)$ or $f'(x)$.

(a) Turn in your programs for Newton’s method on $f(x) = x^2 - 8$.

(b) Write down your results when $p_0 = 2$ and $N = 5, 10, 20$ and when $p_0 = 3$ and $N = 5, 10, 20$. 