Homework #2

1. Let

\[ f(x) = \begin{cases} \sqrt{x}, & \text{if } x \geq 0 \\ -\sqrt{-x}, & \text{if } x < 0 \end{cases} \]

(a) Generate Newton’s method’s \( x_1, x_2, x_3 \) in terms of \( x_0 > 0 \). Then guess the value of \( x_n \) for \( n \) even and for \( n \) odd.

(b) Draw the graphical interpretation of Newton’s method in the previous part.

(c) Will Newton’s method converge for any \( x_0 \neq 0 \)? Why does this not violate the theorem on the convergence of Newton’s method (Theorem 1 on page 85) since the initial guess can be arbitrarily close to the root?

2. Let \( \{x_n\}_{n=0}^{\infty} \) be a sequence of approximations, and suppose there is an interval of the form \([r - \alpha, r + \alpha]\), for some \( \alpha > 0 \), and a constant \( K < 1 \) such that when \( x_n \in [r - \alpha, r + \alpha] \), we get \( |x_{n+1} - r| \leq K|x_n - r| \).

(a) Given \( x_0 \in [r - \alpha, r + \alpha] \), why is \( x_n \in [r - \alpha, r + \alpha] \), for all \( n \geq 0 \)?

(b) For any \( x_0 \in [r - \alpha, r + \alpha] \), prove \( x_n \) converges to \( r \).

(c) If \( \{x_n\}_{n=0}^{\infty} \) comes from Newton’s method, we know

\[ x_{n+1} - r = \frac{f''(\xi_n)}{2f'(x_n)}(x_n - r)^2. \]

For the case \( f(x) = x^2 - 2 \), for the root of interest \( r = \sqrt{2} \), and given \( K < 1 \), find the largest \( \alpha > 0 \) satisfying: \( |x_{n+1} - r| \leq K|x_n - r| \), whenever \( x_n \in [r - \alpha, r + \alpha] \).

(d) Conclude that the sequence of approximations generated by Newton’s method, using initial guess \( x_0 = 1 \), will converge.

3. Let \( f \in C^2[a, r] \), where \( r \) is a root of \( f \) and \( a < r \). Furthermore, suppose \( f''(x) < 0 \) in \([a, r]\) and \( f'(r) > 0 \). Prove Newton’s method’s sequence of approximations converges to \( r \) for all initial guesses \( x_0 \in [a, r] \).

4. Consider the problem of finding the point on the graph of \( y = x^3 \) closest to the point \((3, -1)\).

(a) Write down the expression for \( d(x) = \) the square of the distance from \((x, x^3)\) to \((3, -1)\).

(b) Minimize \( d(x) \) by finding the, in this case, unique critical point: approximating the solution of \( f(x) = d'(x) = 0 \) using Newton’s method to generate \( x_3 \) when \( x_0 = 2 \).

5. Consider the equation \( xy^2 + \tan y = x \) that implicitly defines \( y \) as a function of \( x \): \( y = y(x) \). Approximate \( y(1) \) with \( y_3 \) generated by Newton’s method using initial guess \( y_0 = 1 \).
6. Perform two iterations of Newton’s method on the following systems.
   
   (a) Starting with (0, 1),
   
   \[
   \begin{align*}
   4x_1^2 - x_2^2 &= 0 \\
   4x_1 x_2^2 - x_1 &= 1.
   \end{align*}
   \]

   (b) Starting with (1, 1),
   
   \[
   \begin{align*}
   xy^2 + x^2 y + x^4 &= 3 \\
   x^3 y^5 - 2x^5 y - x^2 &= -2.
   \end{align*}
   \]

7. Consider \( f(x) = x(1 - e^x) \), with root \( r = 0 \).
   
   (a) Verify that \( f(0) = 0 \) and \( f'(0) = 0 \) and \( f''(0) \neq 0 \).
   
   (b) Using Newton’s method with \( x_0 = 0.1 \), compute \( |e_{n+1}/e_n| \) for \( n = 0, 1, 2 \) (\( e_n = x_n - r \)). Does this sequence look bounded? What about \( |e_{n+1}/e_n|^2 \) for \( n = 0, 1, 2 \)?
   
   (c) Simplify \( g(x) = f(x)/f'(x) \) and verify \( r = 0 \) is still a root (find limit as \( x \to 0 \)).
   
   (d) Apply Newton’s method with \( x_0 = 0.1 \) on \( g(x) \), and study \( |e_{n+1}/e_n| \) for \( n = 0, 1 \). Does this sequence look bounded?
   
   (e) Apply the iterative method
   
   \[ x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)}, \]

   for \( m = 2 \), using \( x_0 = 0.1 \), and study \( |e_{n+1}/e_n| \) for \( n = 0, 1 \). Does this sequence look bounded?

8. (Matlab) Suppose we have two Matlab functions, in “hw2f.m” and “hw2fprime.m”, that both take as input \( x \) and output expressions for \( f(x) \) and \( f'(x) \), respectively. Then write a Matlab function that inputs
   
   • initial guess \( x_0 \);
   
   • tolerance \( tol \);

   and outputs the first \( N \), and \( x_N \) of Newton’s method, such that \( |f(x_N)| < tol \). Make sure you call “hw2f” and “hw2fprime” when you need values of \( f(x) \) or \( f'(x) \).
   
   (a) Write out or print out your function. for Newton’s method on \( f(x) = x^2 - 8 \).
   
   (b) For \( f(x) = x^2 - 8 \), write out or print out your results when \( x_0 = 2 \) and \( tol = 10^{-11} \).
   
   Do the same for \( x_0 = 10 \) and \( tol = 10^{-11} \).
   
   (c) For \( f(x) = x^2 \), write out or print out your results with when \( x_0 = 1 \) and \( tol = 10^{-11} \).