Homework #3

1. (a) Use the secant method to generate the approximations $x_2, x_3, x_4$ of $2\sqrt{2}$ by finding the positive root of $x^2 - 8$ using $x_0 = 2, x_1 = 3$. Also write down the absolute error of your final approximation.

(b) Consider the following variation of the bisection method: for each step, with bracketing interval $[a, b]$, approximations and interval cut location are both chosen at $b - \frac{f(b)(b-a)}{f(b)-f(a)}$. We call this variation the method of false position. Use this method to generate approximations $c_0, c_1, c_2$ of $2\sqrt{2}$ by finding the positive root of $x^2 - 8$ using starting interval $[a_0, b_0]$, with $a_0 = 2, b_0 = 3$. Also write down the absolute error of your final approximation.

2. (a) Show the linear system $Ax = \vec{b}$ is equivalent to the fixed point problem $\vec{x} = D^{-1}[(E + F)\vec{x} + \vec{b}]$, where $A = D - E - F$ with $D$ a diagonal matrix, $E$ strictly lower triangular, and $F$ strictly upper triangular.

(b) Find $\vec{x}^{(2)}$ of fixed point iterations, given $\vec{x}^{(0)} = [0, 0]^T$ when $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $\vec{b} = [-5, 7]^T$.

3. Consider fixed point function $F(x) = 1/x + x/2$ and consider the interval $[1.4, 1.45]$.

(a) Show $F(x) \in [1.4, 1.45]$ for $x \in [1.4, 1.45]$ by finding the maximum value, $M$, and minimum value, $m$, of $F$ in the interval and showing $[m, M] \subseteq [1.4, 1.45]$.

(b) Find $\lambda < 1$ such that $|F'(x)| \leq \lambda$ for all $x \in [1.4, 1.45]$ by finding the maximum value of $|F'(x)|$ in the interval.

(c) Conclude that fixed point iterations converge for all $x_0 \in [1.4, 1.45]$.

4. Let $F$ be a continuously differentiable function, and let $s$ be a fixed point of $F$.

(a) Prove if $|F'(s)| < 1$, then there exists $\alpha > 0$ such that fixed point iterations will converge to $s$ whenever $x_0 \in [s - \alpha, s + \alpha]$.

(b) Prove if $|F'(s)| > 1$, then given fixed point iterations $x_n$ satisfying $x_n \neq s$ for all $n$, $x_n$ will not converge to $s$.

5. Let $f(x) = x^3 - 2x + 1$. To solve $f(x) = 0$, consider the three fixed point functions

\[ F_1(x) = \frac{1}{2}(x^3 + 1), \quad F_2(x) = \frac{2}{x} - \frac{1}{x^2}, \quad F_3(x) = \sqrt[3]{2 - \frac{1}{x}}. \]

(a) Verify $r = 1$ is a root of $f$, and also a fixed point for each of the fixed point functions.

(b) Compute $F_1'(1), F_2'(1), F_3'(1)$ and comment whether fixed point iterations will converge for all initial guesses $x_0$ sufficiently close to $r$. For the ones that converge, find the order of convergence.
6. Let \( a < b \), and suppose \( f \in C^2[a, b] \) satisfies \( f'(x) > 0 \) and \( f''(x) > 0 \) for all \( x \in [a, b] \). Also suppose \( f(a) < 0 \) and \( f(b) > 0 \).

(a) Prove there exists a unique root of \( f \) in \((a, b)\), call it \( r \).

(b) Let the approximations \( c_n \) be generated from the method of false position using starting interval \([a, b]\). Prove \( c_0 < r \) and \( f(c_0) < 0 \). Does this hold for all \( c_n \)?

(c) Conclude that \( c_{n+1} = c_n - \frac{f(c_n)(c_n - b)}{f(c_n) - f(b)} \), and \( c_{n+1} > c_n \). Thus \( c_n \) is an increasing sequence bounded above by \( r \), and has to converge to some \( p \in [a, b] \).

(d) Consider the fixed point function \( F(x) = x - \frac{f(x)(x-b)}{f(x) - f(b)} \), so \( c_{n+1} = F(c_n) \). Prove \( c_n \to r \). Then, calculate \( F'(r) \) and comment on the order of convergence of the method of false position in this setting.

7. Answer True or False for the following questions (you do not need to show work but write down your explanation if you are unsure). For each question, \( f \in C^2(\mathbb{R}) \).

(a) Newton’s method gets the exact root after one iteration, for any initial guess, when \( f \) is a line with nonzero slope.

(b) When the function \( f(x) \) has multiple roots, when Newton’s method converges, it always converges to the root closest to the initial guess used.

(c) If the initial guesses satisfy \( x_0 = a, \ x_1 = b \), with \( a < b \), then if secant method converges, it has to converge to a point in \([a, b]\).

(d) Using the same initial guesses, secant method on \( f(x) \) and \(-3f(x)\) produce the same sequence of approximations.

(e) For \( f(x) = x^k, \ k > 1, \) which has root at 0, Newton’s method, for starting guess \( x_0 \neq 0 \), has order of convergence 1.

8. (Matlab) Suppose we have a Matlab function, in “hw3f.m”, that takes as input \( x \) and outputs the expression for \( f(x) \). Write a Matlab function that inputs

- initial guesses \( x_0, \ x_1; \)
- tolerance \( tol; \)

and outputs the first \( N \geq 1 \) and \( x_N \) of secant method such that \( |x_N - x_{N-1}| < tol \). Make sure you call “hw3f” when you need values of \( f(x) \).

(a) Write out or print out your function.

(b) For \( f(x) = \cos x - x \), write out or print out your results when \( x_0 = 0, \ x_1 = \pi/2 \), and \( tol = 10^{-7} \).