Homework #3

1. (b) For the first computation, we have

\[ c_0 = 3 - \frac{f(3)(3-2)}{f(3) - f(2)} = 3 - \frac{1}{5} = \frac{14}{5} = 2.8, \]

where \( f(c_0) = -0.16 < 0 \). All the computations are summarized in the following table:

<table>
<thead>
<tr>
<th>a</th>
<th>c</th>
<th>b</th>
<th>f(a)</th>
<th>f(c)</th>
<th>f(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2.8</td>
<td>3</td>
<td>-4</td>
<td>-0.16</td>
<td>1</td>
</tr>
<tr>
<td>2.8</td>
<td>2.8276</td>
<td>3</td>
<td>-0.16</td>
<td>-0.0047562</td>
<td>1</td>
</tr>
<tr>
<td>2.8276</td>
<td>2.8284</td>
<td>3</td>
<td>-0.0047562</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

2. (b) Note,

\[ D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}. \]

So, since \( \vec{x} = [0, 0]^T \),

\[ \vec{x}^{(1)} = D^{-1}[(E + F)\vec{x}^{(0)} + \vec{b}] = D^{-1}\vec{b} = \begin{bmatrix} -5/2 \\ 7/2 \end{bmatrix}. = \begin{bmatrix} -2.5 \\ 3.5 \end{bmatrix}. \]

Then

\[ \vec{x}^{(2)} = D^{-1}[(E + F)\vec{x}^{(1)} + \vec{b}] = D^{-1} \left( \begin{bmatrix} 7/2 \\ -5/2 \end{bmatrix} + \begin{bmatrix} -5 \\ 7 \end{bmatrix} \right) = \begin{bmatrix} -3/4 \\ 9/4 \end{bmatrix} = \begin{bmatrix} -0.75 \\ 2.25 \end{bmatrix}. \]

3. (b) \( F'(x) = -1/x^2 + 1/2 \). For the maximum of \( F'(x) \), we check through the critical points of \( F'(x) \) and the endpoints. Now

\[ F''(x) = \frac{2}{x^3}, \]

so there are no solutions to \( F''(x) = 0 \), and so there are no critical points of \( F'(x) \) in \([1.4, 1.45]\). This leaves endpoints, and we see \( |F'(1.4)| = 0.010204 \) and \( |F'(1.45)| = 0.024376 \). The largest among these is the maximum of \( |F'(x)| \) in \([1.4, 1.45]\), so

\[ \max_{x \in [1.4,1.45]} |F'(x)| = 0.024376. \]

We can take \( \lambda = 0.024376 < 1 \).

4. (a) Since \( |F'(s)| < 1 \), given \( |F'(s)| < \lambda < 1 \), \( F' \) continuous implies there exists \( \alpha > 0 \) such that \( |x - s| \leq \alpha \) implies \( |F'(x)| \leq \lambda \) (for a \( \epsilon-\delta \) proof, you can use \( \epsilon = \lambda - |F'(s)| \) and \( \alpha = \delta \)). Note \( |x - s| \leq \alpha \) is the same as \( x \in [s - \alpha, s + \alpha] \).

So for \( x, y \in [s - \alpha, s + \alpha] \), the Mean Value Theorem says \( F(x) - F(y) = F'(\xi)(x - y) \), for some \( \xi \) between \( x, y \). This means

\[ |F(x) - F(y)| = |F'(\xi)||x - y| \leq \lambda|x - y|, \]
and so $F$ is a contractive map in $[s - \alpha, s + \alpha]$.
Also, note for $x \in [s - \alpha, s + \alpha],$
\[
|F(x) - s| = |F(x) - F(s)| \leq \lambda |x - s| < |x - s| \leq \alpha,
\]
which implies $F(x) \in [s - \alpha, s + \alpha]$. This means $F : [s - \alpha, s + \alpha] \rightarrow [s - \alpha, s + \alpha]$.
The fixed point theorem then says $F : [s - \alpha, s + \alpha] \rightarrow [s - \alpha, s + \alpha]$ and $F$ a contractive map in $[s - \alpha, s + \alpha]$ implies fixed point iterations converge to $s$ for any starting approximation $x_0 \in [s - \alpha, s + \alpha]$.

6. (d) By the previous part, $c_n$ is increasing and converging to some $p \leq r < b$. Note $F$ is continuous at $r$, since $f$ is continuous and $f(r) - f(b) \neq 0$. So
\[
p = F(p) = p - \frac{f(p)(p-b)}{f(p) - f(b)},
\]
which implies $f(p) = 0$. By the uniqueness of roots in $(a, b)$, this means $p = r$, and $c_n \rightarrow r$.
Now,
\[
F'(x) = 1 - \frac{f'(x)(x-b)}{f(x) - f(b)} + \frac{f(x)(x-b)}{(f(x) - f(b))^2} f'(x),
\]
so
\[
F'(r) = 1 + \frac{f'(r)(r-b)}{f(b)}.
\]
Note, $f(b) = f(r) + f'(r)(b-r) + (f''(\xi)/2)(b-r)^2$, for some $\xi$ between $b, r$, so
\[
F'(r) = 1 + \frac{f'(r)(r-b)}{f'(r)(b-r) + (f''(\xi)/2)(b-r)^2}
= 1 - \frac{1}{1 + (f''(\xi)/(2f'(r)))(b-r)}.
\]
But, $f''(\xi) > 0$ and $f'(r) > 0$ and $b-r > 0$, so
\[
F'(r) = 1 - \frac{1}{1 + A},
\]
for $A > 0$, and $0 < 1/(1 + A) < 1$, which implies $0 < F'(r) < 1$. Since $F'(r) \neq 0$, we have order of convergence 1.

8. (Matlab)
(a) See “hw3afn.m”.
(b) $N = 7$ and $x_7 = 0.73909$. 

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