1. Consider the polynomial $p(x) = 2x^3 - 12x^2 + 22x - 12$.
   
   (a) Verify $x = 1, 2, 3$ are roots of $p$.
   
   (b) Suppose a method is used and approximates a root of $2.1$. Use polynomial division to find $q(x)$ and constant $R$ such that $p(x) = (x - 2.1)q(x) + R$.
   
   (c) Use the quadratic formula to find the roots of $q(x)$.

2. In each part, for given data points $(-1, 2), (1, 3), (2, -2)$, write down (but do not solve) a linear system, enforcing $p(-1) = 2, p(1) = 3, p(2) = -2$, for the unknown coefficients $a, b, c$:
   
   (a) $p(x) = ax^2 + bx + c$.
   
   (b) $p(x) = a\frac{(x-1)(x-2)}{6} + b\frac{(x+1)(x-2)}{2} + c\frac{(x+1)(x-1)}{3}$ (Lagrange form).
   
   (c) $p(x) = a + b(x + 1) + c(x + 1)(x - 1)$ (Newton form).

3. Consider the data points $(-2, -1), (0, 1), (-1, 3)$.
   
   (a) Write down the Lagrange form for the interpolation polynomial $p_2(x)$ for these data points.
   
   (b) (not due) Find $K, a, b, c$ such that $p_3(x) = p_2(x) + K(x-a)(x-b)(x-c)$ interpolates, in addition, the data point $(1, -1)$.
   
   (c) (not due) Evaluate $p_3(-0.7)$ and $p_3'(-0.7)$.

4. (a) Given $n + 1$ data points with distinct nodes, prove there are an infinite number of different polynomials of degree $n + 1$ that interpolate these data points. (Try adding a data point)
   
   (b) (not due) What about polynomials of degree $p > n + 1$?
   
   (c) (not due) Write down two different polynomials of degree exactly 3 interpolating the data points $(1, 4), (2, -2), (3, 1)$.

5. (a) Find the Lagrange form of the interpolation polynomial for the data points $(x_0 - h, f(x_0 - h)), (x_0, f(x_0)), (x_0 + h, f(x_0 + h))$, where $h > 0$, and call it $p_2(x)$.
   
   (b) Simplify $p_2'(x_0)$ into the form $Af(x_0 + h) + Bf(x_0) + Cf(x_0 - h)$, for some $A, B, C$ constants that depend on $h$.
   
   (c) Do the same for $p_2''(x_0)$.
   
   (d) (not due) Do the same for $\int_{x_0 - h}^{x_0 + h} p_2(x) \, dx$. 


6. (not due) Consider data points \((x_i, f(x_i)), i = 0, \ldots, n\) and let \(P(x) = a_0 + a_1 x + \ldots + a_n x^n\) be the interpolation polynomial for these data points. Using the Lagrange form of \(P(x)\) show

\[ a_n = \sum_{i=0}^{n} \prod_{j=0, j \neq i}^{n} \frac{1}{x_i - x_j}. \]

7. Consider data points \((x_i, f(x_i)), i = 0, \ldots, n\), and let \(P(x)\) be the interpolation polynomial for the data. Suppose \(Q(x)\) is the interpolation polynomial for the data points \((x_i, f(x_i)), i = 0, \ldots, n-1\), and \(R(x)\) is the interpolation polynomial for the data points \((x_i, f(x_i)), i = 1, \ldots, n\). Prove

\[ P(x) = \frac{(x - x_0)R(x) - (x - x_n)Q(x)}{x_n - x_0}. \]

8. (not due) Let \(f(x) = e^x\) be the underlying function in the interval \([0.1, 0.6]\), and let \(P(x)\) be the interpolation polynomial for the data points \((0.1, f(0.1)), (0.5, f(0.5)), (0.6, f(0.6))\).

(a) Find \(P(x)\) in Lagrange form.
(b) Use calculus to find the maximum of \(|f'''(x)|\) in the interval \([0.1, 0.6]\).
(c) Use this result to find a constant \(A\) such that \(|f(0.2) - P(0.2)| \leq A\).
(d) Now use calculus to find the maximum of \(|(x - 0.1)(x - 0.5)(x - 0.6)|\) in the interval \([0.1, 0.6]\).
(e) Use your results to find a constant \(B\) such that \(|f(x) - P(x)| \leq B\), for all \(x \in [0.1, 0.6]\).

9. (Matlab) Suppose we have a Matlab function, in “hw4f.m”, that takes as input \(x\) and outputs the expression for \(f(x)\). Write a Matlab function that inputs

- integer \(m\);
- two vectors of data points \(x\) and \(y\), both with \(m\) components;
- location \(z\);

and uses the Lagrange form to output the interpolation polynomial’s value at \(z\). Make sure you call “hw4f” when you need values of \(f(x)\).

(a) Write out or print out your program.
(b) Apply your program to the case with data points \((x_i, f(x_i)), i = 0, \ldots, 20\), where the \(f(x) = \frac{1}{1+25x^2}\), and \(x_i\) are equally spaced nodes satisfying

\[-1 = x_0 < x_1 < \ldots < x_{20} = 1,\]

and write out or print out your results for \(z = 0.975\) (note the exact value is 
\(f(0.975) = 0.040379\)).
(c) (not due) Do the same with the same \(f(x)\), but equally spaced nodes

\[-1 = x_0 < x_1 < \ldots < x_{40} = 1.\]