Homework #4

1. Consider the fixed point function $G = (g_1, g_2)$, where

\[ g_1(x_1, x_2) = \frac{x_1^2 + x_2^2 + 8}{10}, \]
\[ g_2(x_1, x_2) = \frac{x_1 x_2^2 + x_1 + 8}{10}. \]

Let $D = \{(x_1, x_2)^t | 0 \leq x_1, x_2 \leq 1.5\}$.

(a) Show $(g_1(x_1, x_2), g_2(x_1, x_2))^t \in D$ for all $(x_1, x_2)^t \in D$.

(b) (not due) Find $K < 1$ such that

\[ \left| \frac{\partial g_i(x_1, x_2)}{\partial x_j} \right| \leq \frac{K}{2}, \]

for all $(x_1, x_2)^t \in D$ and $i, j \in \{1, 2\}$.

(c) (not due) Perform three iterations of fixed point iterations on $G$ with initial guess $(0.9, 0.9)^t$.

2. Consider the nonlinear system

\[ \begin{cases} x_1(1 - x_1) + 4x_2 - 12 = 0, \\ (x_1 - 2)^2 + (2x_2 - 3)^2 - 25 = 0. \end{cases} \]

Perform three iterations of Newton’s method with initial guess $(0, 0)^t$ to find an approximate solution to the nonlinear system.

3. For each part, find the interpolating polynomial for the data points

$(-1, 2), (1, 3), (2, -2)$.

by writing down a linear system involving $p(-1) = 2, p(1) = 3, p(2) = -2$ and solving it (using Matlab or any method) for the unknown coefficients $a, b, c$:

(a) $p(x) = ax^2 + bx + c$.

(b) $p(x) = a\frac{(x-1)(x-2)}{6} + b\frac{(x+1)(x-2)}{-2} + c\frac{(x+1)(x-1)}{3}$ (Lagrange form).

(c) (not due) $p(x) = a + b(x + 1) + c(x + 1)(x - 1)$ (Newton form).

(d) (not due) Simplify each and show they are all the same polynomial.

4. Consider the data points $(-2, -1), (0, 1), (-1, 3)$.

(a) Write down the Lagrange form for the least degree polynomial $p_2(x)$ interpolating these data points.
(b) (not due) Find $K, a, b, c$ such that $p_3(x) = p_2(x) + K(x - a)(x - b)(x - c)$ interpolates, in addition, the data point $(1, -1)$.

(c) (not due) Evaluate $p_3(-0.7)$ and $p'_3(-0.7)$.

5. (a) Given $n + 1$ data points with distinct nodes, prove there are an infinite number of different polynomials of degree exactly $n + 1$ interpolating these data points (Hint: add a data point).

(b) (not due) Write down two different polynomials of degree exactly 3 interpolating the data points

$$(1, 4), (2, -2), (3, 1).$$

6. (a) Find the Lagrange form of the Lagrange interpolating polynomial for the data points $(x_0 - h, f(x_0 - h)), (x_0, f(x_0)), (x_0 + h, f(x_0 + h))$ and call it $p_2(x)$.

(b) Simplify $p'_2(x_0)$ into the form $Af(x_0 + h) + Bf(x_0) + Cf(x_0 - h)$, for some $A, B, C$ constants that depend on $h$.

(c) Do the same for $p''_2(x_0)$.

(d) (not due) Do the same for $\int_{x_0-h}^{x_0+h} p_2(x) \, dx$.

7. Prove the Lagrange interpolating polynomial, interpolating $n + 1$ data points with distinct nodes, is also the polynomial of least degree that interpolates those data points.

8. (Matlab) Write a Matlab program that inputs

- integer $m$;
- two vectors of data points $x$ and $y$, both with $m$ components;
- location $z$.

Have this program use the Lagrange form for the Lagrange interpolating polynomial to output the polynomial’s value at $z$.

(a) Write out or print out your program.

(b) Apply your program to the case with data points $(x_i, f(x_i)), i = 0, \ldots, 20$, where the $f(x) = \frac{1}{1+25x^2}$, and $x_i$ are equally spaced nodes satisfying

$$-1 = x_0 < x_1 < \ldots < x_{20} = 1,$$

and write out or print out your results for $z = 0.975$, along with the absolute error (exact is $f(z)$).

(c) Do the same with the same $f(x)$, but equally spaced nodes

$$-1 = x_0 < x_1 < \ldots < x_{40} = 1.$$