Homework #4

1. (a) For each \( g_i(x_1, x_2) \), check critical points and boundary values for \( D \) to find the maximum and minimum values, and make sure these are \( \geq 0 \) and \( \leq 1.5 \).

(b) For each \( i, j \), maximize \( |\partial g_i/\partial x_j| \) by checking critical points and boundary values of \( \partial g_i/\partial x_j \) for \( D \).

2. 

\[
\begin{align*}
f_1(x_1, x_2) &= x_1(1 - x_1) + 4x_2 - 12 \\
f_2(x_1, x_2) &= (x_1 - 2)^2 + (2x_2 - 3)^2 - 25,
\end{align*}
\]

and so the Jacobian is

\[
\begin{bmatrix}
1 - 2x_1 & 4 \\
2(x_1 - 2) & 4(2x_2 - 3)
\end{bmatrix},
\]

and its inverse is

\[
\frac{1}{4(1 - 2x_1)(2x_2 - 3) - 8(x_1 - 2)} \begin{bmatrix}
4(2x_2 - 3) & -4 \\
-2(x_1 - 2) & 1 - 2x_1
\end{bmatrix}.
\]

Thus

\[
\bar{x}^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} -12 & -4 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} -12 \\ -12 \end{bmatrix} = \begin{bmatrix} 48 \\ -15 \end{bmatrix},
\]

and

\[
\bar{x}^{(2)} = \begin{bmatrix} -23.94262596542847, 7.60867966164031 \\ 4.16314669044419 \end{bmatrix},
\]

\[
\bar{x}^{(3)} = \begin{bmatrix} -11.8216006270872 \\ 4.16314669044419 \end{bmatrix}.
\]

3. (a) \( p(-1) = 2, p(1) = 3, p(2) = -2 \) gives

\[
\begin{cases}
a - b + c = 2 \\
a + b + c = 3 \\
4a + 2b + c = -2.
\end{cases}
\]

Solving the linear system using Matlab, we get \( a = -11/6, b = 1/2, c = 13/3 \).

(b) Linear system should use the identity matrix.

(c) Linear system should use a lower triangular matrix.

4. (a) Use the Lagrange form.

(b) \(-2, 0, -1\) should be roots of \( K(x - a)(x - b)(x - c) \), and once choice is \( a = -2, b = 0, c = -1 \). Then solve for \( K \) through \( P(1) = -1 \).
5. (a) Let \((x_i, y_i), i = 0, \ldots , n\) be the data points with distinct nodes, and fix \(z\) distinct from all the \(x_i\) (for example, \(z = \min_{0 \leq i \leq n} x_i - 1\)). Now let \(P_w\), for \(w\) a real number, denote the Lagrange interpolating polynomial interpolating the data points \((x_i, y_i), i = 0, \ldots , n\) and, in addition, the data point \((z, w)\). Then \(P_w\) is different, for different \(w\), since \(P_{w_1}(z) \neq P_{w_2}(z)\), for \(w_1 \neq w_2\). Furthermore, \(\text{deg} P_w \leq n + 1\).

Now suppose \(\text{deg} P_{w_1} \leq n\) and \(\text{deg} P_{w_2} \leq n\). Then, since \(P_{w_1}\) and \(P_{w_2}\) interpolate the \(n + 1\) data points \((x_i, y_i), i = 0, \ldots , n\), and there is a unique degree \(\leq n\) polynomial that does this, \(P_{w_1} \equiv P_{w_2}\). Thus, there is at most one polynomial, of the form \(P_w\), with degree \(\leq n\), and the rest, infinitely many (since we can choose, for example, \(w = 0, 1, 2, \ldots\)), have degree \(= n + 1\).

(b) Use the procedure in part (a).

6. (a) Use Lagrange form to write out an expression.

(b) \[
p_2(x) = f(x_0 - h)\frac{(x - x_0)(x - x_0 - h)}{(-h)(-2h)} + f(x_0)\frac{(x - x_0 + h)(x - x_0 - h)}{h(-h)} + f(x_0 + h)\frac{(x - x_0 + h)(x - x_0)}{2h(h)},
\]

so \[
p'_2(x_0) = f(x_0 - h)\frac{-h}{2h^2} + f(x_0)\frac{-h + h}{-h^2} + f(x_0 + h)\frac{h}{2h^2}
 = f(x_0 - h)\frac{-1}{2h} + f(x_0 + h)\frac{1}{2h},
\]

so \(A = 1/(2h)\), \(B = 0\), and \(C = -1/(2h)\).

(c) Should get \(A = 1/h^2\), \(B = -2/h^2\), and \(C = 1/h^2\).

(d) Should get \(A = h/3\), \(B = 4h/3\), and \(C = h/3\).

7. The Lagrange interpolating polynomial, \(P(x)\), for the \(n + 1\) data points exists. Thus there exists also a polynomial of least degree, \(Q(x)\), where \(\text{deg} Q \leq \text{deg} P \leq n\), that interpolates the \(n + 1\) data points. However, there is a unique polynomial of degree \(\leq n\) that that interpolates the \(n + 1\) data points, so \(P \equiv Q\).

8. (Matlab)

(a) See “hw4afn.m”.

(b) In this case, \(m = 21\), and we get the approximation \(-59.7819301618324\).

(c) In this case, \(m = 41\), and we get the approximation \(-57409.1797426493\).