**Homework #5**

1. (not due) Consider data points \((x_i, f(x_i)), i = 0, \ldots, n\). Let \(P(x)\) be the Lagrange interpolating polynomial for these data points. Use the Lagrange form of \(P(x)\) to find an expression for \(a_n\), where \(P(x) = a_0 + a_1 x + \ldots + a_n x^n\).

2. Consider data points \((x_i, f(x_i)), i = 0, \ldots, n\), and let \(P(x)\) be its Lagrange interpolating polynomial. Suppose \(Q(x)\) is the Lagrange interpolating polynomial for the data points \((x_i, f(x_i)), i = 0, \ldots, n-1\), and \(R(x)\) is the Lagrange interpolating polynomial for the data points \((x_i, f(x_i)), i = 1, \ldots, n\). Prove

\[
P(x) = \frac{(x - x_0)R(x) - (x - x_n)Q(x)}{x_n - x_0}.
\]

3. Let \(f(x) = e^x\) be the underlying function in the interval \([0.1, 0.6]\), and let \(P(x)\) be the interpolating polynomial for the data points \((0.1, f(0.1)), (0.5, f(0.5)), (0.6, f(0.6))\).
   
   (a) Find \(P(x)\) in Lagrange form.
   
   (b) Use calculus to find the maximum of \(|(x - 0.1)(x - 0.5)(x - 0.6)|\) in the interval \([0.1, 0.6]\).
   
   (c) Use calculus to find the maximum of \(|f'''(x)|\) in the interval \([0.1, 0.6]\).
   
   (d) Use these results to bound \(|f(x) - P(x)|\) in the interval \([0.1, 0.6]\).
   
   (e) (not due) Verify the exact absolute errors at the points \(x = 0.2, 0.3, 0.4\) satisfy these bounds.

4. (a) Draw a graph of the piecewise linear interpolating polynomial for the data given in the following table:

\[
\begin{array}{c|cccc}
  x & 0 & 1 & 3 & 4 & 6 \\
  f(x) & 2 & 4 & 1 & -2 & -3 \\
\end{array}
\]

   (b) Write down the equation for the linear piece in the interval \([4, 6]\). Then use it to approximate \(f'(4), f'(6), \) and \(f_4^6 f(x) \, dx\).

   (c) Suppose instead we create the piecewise quadratic interpolating polynomial by using the parabola passing through the first three points \(x \in [0, 3]\) and the parabola passing through the last three points \(x \in [3, 6]\). Find the values of this piecewise quadratic interpolant at \(x = 1.5\) and also at \(x = 5.7\).

5. Let \(f(x) = \sin x\) in the interval \([-\pi, \pi]\). Let \(P(x)\) be the piecewise linear interpolating polynomial at \(n + 1\) evenly spaced data points, \((x_i, f(x_i)),\) starting from \(x_0 = -\pi\) and ending at \(x_n = \pi\). Consider \(n = 100\).

   (a) Use the error bound

\[
|f(x) - P(x)| \leq \frac{h^2}{8} \max_{-\pi \leq c \leq \pi} |f''(c)|
\]

   to bound \(|f(1) - P(1)|\).
(b) Verify the actual absolute error, $|f(1) - P(1)|$, satisfies this bound.

6. Use error bound and calculus to find the constant $C$ such that

$$|f(x) - P(x)| \leq C$$

for all $x \in [0, 1]$, where $P(x)$ is the piecewise linear interpolating polynomial for data with nodes $x_j = j/10, j = 0, \ldots, 10$ and values from the underlying function $f(x) = x^2 + 1$.

7. Use error bound and calculus to find $n$ ensuring that the piecewise linear interpolating polynomial for data with nodes $x_j = j/n, j = 0, \ldots, n$ and values from the underlying function $f(x) = x^2 + 1$ satisfies

$$|f(x) - P(x)| \leq 10^{-7}$$

for all $x \in [0, 1]$.

8. (Matlab) Write a Matlab program that inputs:
   - $m$;
   - vectors $x$ and $y$, the $x$ and $y$ coordinates of $m$ data points, where the elements of $x$ are in increasing order;
   - location $z$ lying between $x_1$ and $x_m$;

and outputs the value of the linear interpolant evaluated at $z$.

   (a) Write out or print out your program.
   (b) Apply your program to the case $m = 101$ and $x_i$ evenly spaced, from $-\pi$ to $\pi$, and $y_i = \sin x_i$, and write out or print out your results when $z = -3, -1, 0.5, 2$.

9. (Matlab) (not due) Write a Matlab program that inputs:
   - $m$;
   - vectors $x$ and $y$, the $x$ and $y$ coordinates of $m$ data points, where the elements of $x$ are evenly spaced and in increasing order;
   - location $z$ satisfying $x_2 < z < x_{m-1}$;

and outputs the value of the Lagrange interpolating polynomial using only the four data points with the four closest nodes to $z$.

   (a) Write out or print out your program.
   (b) Apply your program to the case $m = 101$ and $x_i$ evenly spaced, from $-\pi$ to $\pi$, and $y_i = \sin x_i$, and write out or print out your results when $z = -3, -1, 0.5, 2$. 

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