Homework #5

1. (a) Draw a graph of the piecewise linear interpolating polynomial for the data given in the following table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>3</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>-2</td>
<td>-3</td>
</tr>
</tbody>
</table>

(b) Write down the equation for the linear piece in the interval $[4, 6]$. Then use it to approximate $f'(4)$, $f'(6)$, and $\int_{4}^{6} f(x) \, dx$.

(c) Suppose instead we create the piecewise quadratic interpolating polynomial by using the parabola passing through the first three points in $x \in [0, 3]$ and the parabola passing through the last three points in $x \in [3, 6]$. Find the values of this piecewise quadratic interpolant at $x = 1.5$ and also at $x = 5.7$.

2. Let $f(x) = \sin x$ in the interval $[-\pi, \pi]$. Let $P(x)$ be the piecewise linear interpolating polynomial at $n + 1$ evenly spaced data points, $(x_i, f(x_i))$, starting from $x_0 = -\pi$ and ending at $x_n = \pi$. Let $h = x_{i+1} - x_i$ and consider $n = 100$.

(a) Find $P(1)$ and the absolute error $|f(1) - P(1)|$.

(b) Use the error bound

$$|f(x) - P(x)| \leq \frac{\max_{z \in [x_i, x_{i+1}]} |f''(z)|}{2} \left|(x - x_i)(x - x_{i+1})\right|$$

with the correct $i$, to bound $|f(1) - P(1)|$. Does the actual value of $|f(1) - P(1)|$ satisfy this bound?

(c) Now for arbitrary $x \in [x_i, x_{i+1}]$, for the same $i$ in the previous part, use the error bound

$$|f(x) - P(x)| \leq \frac{\max_{z \in [x_i, x_{i+1}]} |f''(z)|}{2} \frac{h^2}{4}$$

to bound $|f(1) - P(1)|$. Does the actual value of $|f(1) - P(1)|$ satisfy this bound?

(d) Now for arbitrary $x \in [-\pi, \pi]$, use the error bound

$$|f(x) - P(x)| \leq \frac{\max_{z \in [-\pi, \pi]} |f''(z)|}{2} \frac{h^2}{4}$$

to bound $|f(x) - P(x)|$. Does the actual value of $|f(1) - P(1)|$ satisfy this bound?

3. Use error bounds to find $n$ ensuring that the piecewise linear interpolating polynomial $P(x)$ for data with nodes $x_j = j/n$, $j = 0, \ldots, n$ and values from the underlying function $f(x) = x^2 + 1$ satisfies $|f(x) - P(x)| \leq 10^{-7}$, for all $x \in [0, 1]$.

4. Consider the interval $[a, b]$, for $a < b$.

(a) Find the interpolation polynomial $q(x)$ for the data points $(a, -1), (b, 1)$. Under $q$, $[a, b]$ is mapped to what interval? Write down the expression for $q^{-1}$ as well.
(b) Find monic polynomial \( S_m \) of degree \( m \), in terms of Chebyshev polynomial \( T_m \) and \( q \), such that
\[
\max_{x \in [a,b]} |r(x)| \geq \max_{x \in [a,b]} |S_m(x)|;
\]
for any monic polynomial \( r \) of degree \( m \). What are the \( m \) roots of \( S_m \), using the expression for the roots of \( T_m \)?

(c) Given \( n+1 \) data points with distinct nodes in \([a,b]\), with interpolation polynomial \( p \), use your results to find \( C \) such that
\[
|f(x) - p(x)| \leq C \frac{\max_{z \in [a,b]} |f^{(n+1)}(z)|}{(n+1)!}.
\]

5.  (a) Find the Lagrange form for the interpolation polynomial \( p(x) \) using 3 optimal node locations, according to Chebyshev polynomials, for the function \( f(x) = \cos(\pi x) \) in \([0,1]\), and evaluate it at \( x = 3/4 \).

(b) For arbitrary \( x \in [0,1] \), use your results from Problem 4c to bound \( |f(3/4) - p(3/4)| \). Does the actual value of \( |f(3/4) - p(3/4)| \) satisfy this bound?

6. Use error bounds to find how many nodes, chosen at optimal node locations, according to Chebyshev polynomials, are needed for the function \( f(x) = e^x \) in \([-1,1]\) to ensure the interpolation polynomial \( p(x) \) satisfies \( |f(x) - p(x)| \leq 10^{-7} \), for all \( x \in [-1,1] \).

7. Let \( p(x) \) be a degree \( n \geq 1 \) monic polynomial satisfying:
   
   - \( \max_{x \in [-1,1]} p(x) = -\min_{x \in [-1,1]} p(x) = M \);
   - there exist \( n + 1 \) distinct locations \( x_0, \ldots, x_n \) such that \( |p(x_i)| = M \), for all \( i = 0, \ldots, n \), and \( p(x_i), p(x_{i+1}) \) have opposite signs, for \( i = 0, \ldots, n - 1 \).

   Prove \( M = 1/2^{n-1} \).

8. (Matlab) Suppose we have a function “hw5f.m” that takes as input \( x \) and outputs the value for a function \( f(x) \). Write a Matlab program that inputs:
   
   - interval \([a,b]\);
   - \( m \), the number of data points with evenly spaced nodes from \( x_0 = a \) to \( x_m = b \), and values from \( f(x) \);
   - location \( z \) satisfying \( x_1 < z < x_{m-1} \), where \( h = (b-a)/(m-1) \);

   and outputs the value of the interpolation polynomial using only the four data points with nodes closest to \( z \).

   (a) Write out or print out your program.

   (b) Apply your program to the case \( m = 101 \) and \( x_i \) evenly spaced, from \(-\pi\) to \( \pi \), and \( y_i = \sin x_i \), and write out or print out your results when \( z = -3, -1, 0.5, 2 \).