Homework #6

1. Consider the data points \((-2, -1), (0, 1), (-1, 3)\).
   
   (a) Compute the divided difference table associated to this data, and write down the resulting Newton form for the interpolation polynomial.
   
   (b) Add the data point \((1, -1)\) to your divided difference table and write down the Newton form for the new interpolation polynomial.
   
   (c) From your divided difference table, also write down the Newton form of the interpolation polynomial for the data points \((0, 1), (-1, 3), (1, -1)\).

2. Suppose \(x_0 = 0, x_1 = 0.4, x_2 = 0.7\) and \(f[x_2] = 6, f[x_1, x_2] = 10, f[x_0, x_1, x_2] = 50/7\). Find the values of \(f[x_0], f[x_1], f[x_0, x_1]\).

3. Suppose \(x_0 = 1, x_1 = 2, x_2 = 3, x_3 = 4\) and \(f[x_3] = -2, f[x_2, x_3] = 1, f[x_1, x_2, x_3] = 6, f[x_0, x_1, x_2, x_3] = 7\). For the following, do not complete the rest of the divided difference table (do not find \(f[x_0, f[x_1], f[x_2], f[x_0, x_1], f[x_1, x_2], f[x_0, x_1, x_2]\)).
   
   (a) Find some Newton form of the interpolation polynomial.
   
   (b) Add the data point \((0, -1)\) and find some Newton’s form for the new interpolation polynomial.

4. Consider an underlying function \(f(x)\) and nodes at the locations of \(x_0 - h, x_0, x_0 + h\), for some \(h > 0\). Assuming \(h\) is small enough so that \(f''(x) \approx f''(y)\), for any \(x, y \in [x_0 - h, x_0 + h]\), use the result: \(f[x_0, x_1, t] = f'[x_1] + f''[x_1] t\), to write down an approximation formula for \(f''(x_0)\) as a linear combination of \(f(x_0 - h), f(x_0), f(x_0 + h)\).

5. Given \(f \in C^\infty[a, b]\) and nodes \(x_0 < x_1 < x_2 < x_3\) in \([a, b]\), let \(p\) be the interpolation polynomial for the table of data:

   \[
   \begin{array}{c|cccc}
   x & x_0 & x_1 & x_2 & x_3 \\
   \hline
   f(x) & f(x_0) & f(x_1) & f(x_2) & f(x_3) \\
   f'(x) & f'(x_1) & f'(x_2) & f'(x_3) \\
   f''(x) & f''(x_1) & & &
   \end{array}
   \]

   Find \(j, m, \) and \(k_i, i = 0, \ldots, 3\) such that

   \[
   f(x) = p(x) + \frac{f^{(j)}(\xi_x)}{m!} \prod_{i=0}^{3} (x - x_i)^{k_i},
   \]

   for some \(\xi_x \in (a, b)\).

6. Consider \(f(x) = \sin x\) and evenly spaced nodes \(0 = x_0 < x_1 < \ldots < x_n = 2\pi\). Let \(P(x)\) be the piecewise cubic interpolant given values and first derivatives of \(f\) at the nodes.
(a) In the case $n = 100$, use calculus and the error formula

$$|f(1) - P(1)| \leq \frac{\max_{z \in [x_i, x_{i+1}]} |f^{(4)}(z)|}{4!} |(1 - x_i)^2(1 - x_{i+1})^2|,$$

where $1 \in [x_i, x_{i+1}]$, to bound the absolute error $|f(1) - P(1)|$.

(b) For arbitrary $x \in [0, 2\pi]$, use error bounds to determine $n$ ensuring that $|f(x) - P(x)| \leq 10^{-10}$.

7. Consider data $x_0 = -1, f(x_0) = 0, f'(x_0) = 1$ and $x_1 = 0, f(x_1) = 1, f'(x_1) = 2$.

(a) Use divided differences to find the interpolation polynomial for this data.

(b) Add the data $x_2 = 1, f(x_2) = 2, f'(x_2) = -1$ and find the resulting interpolation polynomial.

(c) Find instead the form for the piecewise cubic interpolant for the three data points.

8. Use divided differences to find the interpolation polynomial for the data

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>-1</td>
<td>-3</td>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>$f'(x)$</td>
<td>1</td>
<td>-2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f''(x)$</td>
<td></td>
<td></td>
<td>-4</td>
<td></td>
</tr>
</tbody>
</table>

9. (Matlab) Write a Matlab program that inputs:

- $m$;
- vectors $x$ and $y$, the $x$ and $y$ coordinates of $m$ data points;

and uses a divided difference table to calculate and output the coefficient of the $x^{m-1}$ term in the interpolation polynomial.

(a) Write out or print out your program.

(b) Apply your program to the case with data points $(x_i, f(x_i)), i = 0, \ldots, 20$, where the $f(x) = \frac{1}{1 + 25x^2}$, and $x_i$ are equally spaced nodes satisfying $-1 = x_0 < x_1 < \ldots < x_{20} = 1$, and write out or print out your results.