Homework #6

1. (a) Picture should be composed of lines in each of the intervals $[0, 1], [1, 3], [3, 4], [4, 6]$.
   (b) The line is $-(x - 4)/2 - 2$, and its derivative is $-1/2$ so we can use this line to approximate $f'(4) \approx -1/2$ and $f'(6) \approx -1/2$. Integrating the line, we get $\int_4^6 f(x) \, dx \approx -5$.
   (c) The quadratic polynomial in $[0, 3]$ is used at location $x = 1.5$, giving value $4.125$.
      The quadratic polynomial in $[3, 6]$ is used at location $x = 5.7$, giving value $-3.275$.

2. Since
   \[ |f(x) - P(x)| \leq \frac{h^2}{4}, \]
   we just force $h^2/4 \leq 10^{-7}$ to get $h \leq 6.3246 \cdot 10^{-4}$.

3. (a) $q(x)$ will be a line, as will $q^{-1}(x)$.
   (b) Note
   \[
   \begin{align*}
   \max_{x \in [a,b]} |r(x)| &= \max_{x \in [-1,1]} |r(q^{-1}(x))| \\
   &= \left(\frac{b-a}{2}\right)^m \max_{x \in [-1,1]} \left| \frac{r(q^{-1}(x))}{\left(\frac{b-a}{2}\right)^m} \right| \\
   &\geq \left(\frac{b-a}{2}\right)^m \max_{x \in [-1,1]} |\tilde{T}_m(x)| \\
   &= \max_{x \in [a,b]} \left| \frac{\tilde{T}_m(q(x))}{\left(\frac{b-a}{2}\right)^m} \right|
   \end{align*}
   \]
   (c) Since $\max_{x \in [-1,1]} |\tilde{T}_m(x)| = \frac{1}{2^{m-1}}$, we can take $C$ to be $\frac{1}{2^n} \left(\frac{b-a}{2}\right)^{n+1}$.

4. (a) Use the roots of $S_3(x)$, when $a = 0, b = 1$, for node locations.
   (b) Can bound
   \[ |f(3/4) - p(3/4)| \leq 0.16149, \]
   and in reality $|f(3/4) - p(3/4)| = 0.14249$.

5. Use the bound
   \[ |f(x) - p(x)| \leq \frac{1}{2^n} \frac{1}{(n+1)!} e, \]
   and force $e/(2^n(n+1)!) \leq 10^{-7}$.

6. First note $M > 0$, else $p(x)$ will not be degree $n \geq 1$. Now from the result of Chebyshev polynomials, $M \geq 1/2^{n-1}$, so it remains to show $M \leq 1/2^{n-1}$. Suppose $M > 1/2^{n-1}$ and consider $p(x) - T_n(x)$. This polynomial has degree $\leq n-1$ and will have opposite signs at $x_i, x_{i+1}$. This means by IVT it has at least $n$ roots. The Fundamental Theorem of Algebra then says $p(x) - T_n(x) = 0$ for all $x$, which is a contradiction.
7. (Matlab) See Matlab solutions.