Homework #6

2. The given divided difference table looks like

\[
\begin{array}{c|cc}
0 & f[x_0] & f[x_0, x_1] \\
0.4 & f[x_1] & 50/7 \\
0.7 & 6 & 10 \\
\end{array}
\]

so
\[
\frac{50}{7} = \frac{10 - f[x_0, x_1]}{0.7},
\]
and \( f[x_0, x_1] = 5 \); and
\[
10 = \frac{6 - f[x_1]}{0.3},
\]
and \( f[x_1] = 3 \); and
\[
5 = \frac{3 - f[x_0]}{0.4},
\]
and \( f[x_0] = 1 \).

3. (a) The divided difference table looks like

\[
\begin{array}{c|ccc}
1 & f[x_0] & f[x_0, x_1] & f[x_0, x_1, x_2] \\
2 & f[x_1] & f[x_1, x_2] & 7 \\
3 & f[x_2] & 6 & 1 \\
4 & 1 & 101/24 & 67/24 \\
0 & -1 & 5/12 & -1/4 \\
\end{array}
\]

so a Newton’s form for the interpolation polynomial can take the form
\[
-2 + (x - 4) + 6(x - 4)(x - 3) + 7(x - 4)(x - 3)(x - 2).
\]

(b) We can add the data point to the divided difference table to get

\[
\begin{array}{c|cccc}
1 & f[x_0] & f[x_0, x_1] & f[x_0, x_1, x_2] & f[x_0, x_1, x_2, x_3] \\
2 & f[x_1] & f[x_1, x_2] & 7 & 101/24 \\
3 & f[x_2] & 6 & 1 & 67/24 \\
4 & 1 & 5/12 & 1/4 & 0 \\
0 & -1 & 1/4 & -1/4 & -1 \\
\end{array}
\]

and so a Newton’s form for the interpolation polynomial can take the form
\[
-1 - \frac{1}{4}x + \frac{5}{12}x(x - 4) + \frac{67}{24}x(x - 4)(x - 3) + \frac{101}{24}x(x - 4)(x - 3)(x - 2).
\]
5. We choose \( j = k = 7 \), since there are 7 total values, including derivative values, given. Since \( x_1 \), value, first derivative, and second derivative are given, we triple that node, and \( m_1 = 3 \). Also at \( x_3 \), value and first derivative are given, so we double that node, and \( m_3 = 2 \). For the rest, \( m_0 = m_2 = 1 \). This leads to our guess for error formula,

\[
\frac{f(x) = P(x) + \frac{f^{(7)}(\xi(x))}{7!} (x-x_0)(x-x_1)^2(x-x_2)(x-x_3)^2}{(x-x_0)(x-x_1)^2(x-x_2)(x-x_3)^2}.
\]

When \( x \) is a node, the error formula is satified as \( f(x) = P(x) \). For \( x \in [a,b] \) not a node, fix it and consider

\[
g(t) = f(t) - P(t) - [f(x) - P(x)] \frac{(t-x_0)(t-x_1)^2(t-x_2)(t-x_3)^2}{(x-x_0)(x-x_1)^2(x-x_2)(x-x_3)^2}.
\]

Now \( g(x_i) = 0 \), for all \( i = 0, 1, 2, 3 \), since \( f(x_i) = P(x_i) \). Also, \( g(x) = f(x) - P(x) - [f(x) - P(x)] = 0 \). So \( g \) has at least 5 roots in \((a,b)\), and by Rolle’s Theorem, \( g' \) has at least 4 roots in \((a,b)\), most notably not at \( x_1 \) and \( x_3 \).

Now since \( f'(x_1) = P'(x_1) \) and \( f'(x_3) = P'(x_3) \), we have \( g'(x_1) = g'(x_3) = 0 \). Thus \( g' \) has at least 6 roots in \((a,b)\), and Rolle’s Theorem says \( g'' \) has at least 5 roots in \((a,b)\), most notably not at \( x_1 \).

Now since \( f''(x_1) = P''(x_1) \), we have \( g''(x_1) = 0 \). Thus \( g'' \) has at least 6 roots in \((a,b)\). Now Generalized Rolle’s Theorem on \( g'' \) says there exists \( \xi(x) \in (a,b) \) such that \( 0 = (g^{(5)}(\xi(x))) = g^{(7)}(\xi(x)) \).

Now since \( \text{deg } P \leq 6 \), \( P^{(7)}(t) = 0 \). Also the seventh derivative of the monic degree 7 polynomial \((t-x_0)(t-x_1)^2(t-x_2)(t-x_3)^2\) is \(7!\). So

\[
0 = g^{(7)}(\xi(x)) = f^{(7)}(\xi(x)) - \frac{f(x) - P(x)}{(x-x_0)(x-x_1)^2(x-x_2)(x-x_3)^2},
\]

and, simplifying, we get

\[
f(x) = P(x) + \frac{f^{(7)}(\xi(x))}{7!} (x-x_0)(x-x_1)^2(x-x_2)(x-x_3)^2.
\]

6. (b) The error bound to use is:

\[
|f(x) - P(x)| = \max_{z \in [0,2\pi]} |f^{(m)}(z)| \left( \frac{h}{2} \right)^4,
\]

since \( |(x-x_i)(x-x_{i+1})|^2 \), for any \( i \), is maximized at the midpoint of \([x_i, x_{i+1}]\), \( x = (x_i + x_{i+1})/2 \), for a maximum of \( (h/2)^4 \). Now

\[
\max_{z \in [0,2\pi]} |f^{(m)}(z)| = \max_{z \in [0,2\pi]} |\sin z| = 1,
\]

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so we force the error bound
\[ \frac{1}{4!} \frac{h^4}{16} \leq 10^{-10}. \]

With \( h = 2\pi/n \), this happens when

\[ n \geq \left( \frac{(2\pi)^4}{384 \cdot 10^{10}} \right)^{1/4} = 448.85. \]

8. We double the node at \( x = 4 \) and triple the node at \( x = 2 \) to form the divided difference table

\[
\begin{array}{cccc}
1 & -1 & -2 \\
2 & -3 & 3 & -5 \\
1 & 2 & 7/2 & -31/24 \\
2 & 0 & -3/8 & -7/144 \\
1 & 5/4 & -23/16 \\
3 & 5/2 & -13/4 \\
6 & -21/4 \\
4 & 4 & -8 & -2 \\
4 & 4 \\
\end{array}
\]

So the interpolation polynomial can take the form

\[
p(x) = -1 - 2(x - 1) + 3(x - 1)(x - 2) - 5(x - 1)(x - 2)^2 + \frac{7}{2}(x - 1)(x - 2)^3 \\
- \frac{31}{24}(x - 1)(x - 2)^3(x - 3) - \frac{7}{144}(x - 1)(x - 2)^3(x - 3)(x - 4). \]

9. (Matlab)

(a) See “hw6afn.m”.

(b) The coefficient is \( 2.6018 \cdot 10^5 \).