Homework #8

1. Determine the best linear least squares approximation for the data

\((-1, 0.03), (-1, -0.05), (0, 1.03), (0.5, 1.48), (1, 1.96)\).

2. Determine the best quadratic least squares approximation for the data

\((-1, 2.04), (-0.5, 1.23), (0, 1.01), (0.5, 1.28), (1, 1.99)\).

3. Derive from minimization the normal equations, in matrix form, for best least squares approximation of data points \((x_1, y_1), \ldots, (x_n, y_n)\) by a quadratic \(a_0 + a_1(1 + x) + a_2(1 + x + x^2)\) under the inner product \(<\vec{v}, \vec{w}> = \sum_{i=1}^{n} c_i v_i w_i\), where \(c_i, i = 1, \ldots, n\) are given positive numbers.

4. (a) Consider the inner product \(<g, h> = \int_{a}^{b} g(x)h(x)w(x) \, dx\), where \(w(x)\) is a given function satisfying \(w(x) > 0\). Now given a function \(f\), derive from minimization the normal equations involved in the best least squares approximation of \(f\) by a line \(a_0 + a_1x\).

(b) Use your results to determine the best linear least squares approximation when \(f(x) = x^2, w(x) = 1, a = 0, b = 1\).

(c) Do this also for \(f(x) = x^2, w(x) = x(1 - x), a = 0, b = 1\).

5. Use the Gram-Schmidt formulas to generate the Legendre polynomials \(P_2(x), P_3(x), P_4(x)\) in \([-1, 1]\), with respect to inner product \(<g, h> = \int_{-1}^{1} g(x)h(x) \, dx\), given \(P_0(x) = 1\) and \(P_1(x) = x\).

6. Let \(f(x) = x^3 + 1\), let \(p_n\) be the Legendre polynomial of degree \(n\), and consider the inner product \(<g, h> = \int_{-1}^{1} g(x)h(x) \, dx\).

(a) Determine the best least squares approximation by a line \(a_0p_0(x) + a_1p_1(x)\) in the interval \([-1, 1]\) under the norm given by the inner product.

(b) Determine the best approximation by a quadratic \(a_0p_0(x) + a_1p_1(x) + a_2p_2(x)\) in the interval \([-1, 1]\) under the norm given by the inner product.

7. (Matlab)

(a) Write a Matlab function that takes as input \(x\) and outputs \(h(x)\). Then write a Matlab function that calls it to output the value of

\[ \sum_{i=0}^{N} c_i h(x_i), \]
where an even \( N \) and an interval \([a, b]\) are given as inputs; \( h = (b - a)/N \) and 
\[ x_i = a + ih; \] and 
\[ c_i = \begin{cases} 
  h/3, & \text{if } i = 0 \text{ or } i = N \\
  4h/3, & \text{if } 0 < i < N \text{ and } i \text{ odd} \\
  2h/3, & \text{if } 0 < i < N \text{ and } i \text{ even.}
\end{cases} \]

Turn in this latter program. The outputed value actually approximates \( \int_a^b h(x) \, dx \).
This will be used in part (b).

(b) Use your program to determine the best approximation of \( f(x) = \cos \pi x \) by \( g \),
where \( g(x) = c_1 p_0(x) + c_2 p_1(x) + c_3 p_2(x) + c_4 p_3(x) + c_5 p_4(x) \), for \( p_n \) the Legendre polynomial of degree \( n \), in the interval \([-1, 1]\) under the norm given by the inner product \( < g, v > = \int_a^b g(x)v(x) \, dx \). Use your program with \( N = 100 \) to compute 
\( < p_n, f > \) whenever needed.