Homework #9

1. (a) Derive from minimization the normal equations for best least squares approximation of data points \((x_1, y_1), \ldots, (x_n, y_n)\) by a line \(a_0 + a_1 x\) under the inner product 
\[ \langle \vec{v}, \vec{w} \rangle = \sum_{i=1}^{n} v_i w_i. \]
(b) Use your results to determine the best linear least squares approximation for the data 
\((-1, 0.03), (-1, -0.05), (0, 1.03), (0.5, 1.48), (1, 1.96).\)

2. (a) Derive from minimization the normal equations for best least squares approximation of data points \((x_1, y_1), \ldots, (x_n, y_n)\) by a quadratic \(a_0 + a_1 x + a_2 x^2\) under the inner product 
\[ \langle \vec{v}, \vec{w} \rangle = \sum_{i=1}^{n} v_i w_i. \]
(b) Use your results to determine the best quadratic least squares approximation for the data 
\((-1, 2.04), (-0.5, 1.23), (0, 1.01), (0.5, 1.28), (1, 1.99).\)

3. (a) Consider the inner product 
\[ \langle g, h \rangle = \int_{a}^{b} g(x) h(x) w(x) \, dx, \] where \(w(x)\) is a given function satisfying \(w(x) > 0\). Now given a function \(f\), derive from minimization the normal equations involved in the best least squares approximation of \(f\) by a line \(a_0 + a_1 x\).
(b) Use your results to determine the best linear least squares approximation when 
\(f(x) = x^2, w(x) = 1, a = 0, b = 1.\)
(c) Do this also for 
\(f(x) = x^2, w(x) = x(1 - x), a = 0, b = 1.\)

4. Let \(G\) be a subspace in an inner product space \(E\).
(a) For \(f \in E\) and \(g \in G\), prove that \(g\) is a best approximation of \(f\) in \(G\) if and only if \(f - g \perp G\). (Hint: Look in the book)
(b) Derive from this the normal equations for best least squares approximation of data points \((x_1, y_1), \ldots, (x_n, y_n)\) by a polynomial of degree \(d\): \(a_0 + a_1 x + \ldots + a_d x^d\) under the inner product 
\[ \langle \vec{v}, \vec{w} \rangle = \sum_{i=1}^{n} v_i w_i. \]
(c) Derive from this the normal equations for best least squares approximation of a function \(f\) by a polynomial of degree \(d\): \(a_0 + a_1 x + \ldots + a_d x^d\) under the inner product 
\[ \langle g, h \rangle = \int_{a}^{b} g(x) h(x) w(x) \, dx, \] where \(w(x)\) is a given function satisfying \(w(x) > 0\).

5. Let \(f(x) = x^3 + 1\), let \(p_n\) be the Legendre polynomial of degree \(n\), and consider the inner product 
\[ \langle g, h \rangle = \int_{-1}^{1} g(x) h(x) \, dx. \]
(a) Determine the best least squares approximation by a line \(a_0 p_0(x) + a_1 p_1(x)\) in the interval \([-1, 1]\) under the norm given by the inner product.
(b) Determine the best approximation by a quadratic \(a_0 p_0(x) + a_1 p_1(x) + a_2 p_2(x)\) in the interval \([-1, 1]\) under the norm given by the inner product.
6. (Matlab)

(a) Write a Matlab function that takes as input $x$ and outputs $h(x)$. Then write a Matlab function that calls it to output the value of

$$\sum_{i=0}^{N} c_i h(x_i),$$

where an even $N$ and an interval $[a, b]$ are given as inputs; $h = (b - a)/N$ and $x_i = a + ih$; and

$$c_i = \begin{cases} 
  h/3, & \text{if } i = 0 \text{ or } i = N \\
  4h/3, & \text{if } 0 < i < N \text{ and } i \text{ odd} \\
  2h/3, & \text{if } 0 < i < N \text{ and } i \text{ even.}
\end{cases}$$

Turn in this latter program. The outputed value actually approximates $\int_{a}^{b} h(x) \, dx$. This will be used in part (b).

(b) Determine the best approximation of $f(x) = \cos \pi x$ by $g$, where $g(x) = c_1 p_0(x) + c_2 p_1(x) + c_3 p_2(x) + c_4 p_3(x) + c_5 p_4(x)$, for $p_n$ the Legendre polynomial of degree $n$, in the interval $[-1, 1]$ under the norm given by the inner product $\langle g, v \rangle = \int_{-1}^{1} g(x)v(x) \, dx$. Use your program with $N = 100$ to compute $\langle p_n, f \rangle$ whenever needed.